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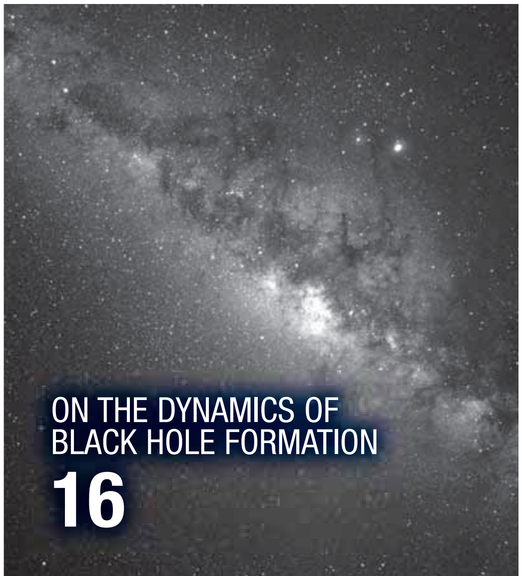

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Canadian Mathematical Society
Société mathématique du Canada

From the Vice-President's Desk

Karl Dilcher, *Dalhousie University*

In praise of older colleagues

All of us who work at a university put a lot of time and effort into teaching and educating graduate students, mentoring them as they take their first steps towards teaching and research, and often taking an active interest in their beginning careers. Postdoctoral fellows are also very

important to our departments; although usually limited to a few years in one place, they enhance a department's strength, invigorate the overall research effort, and provide much needed sessional teaching. The importance of providing the best possible environment and support systems for our young people is well understood, and such support is put in place by programs for graduate students and postdoctoral programs offered by all the institutes, from AARMS to PIMS. All this is essential for the future of our profession, and more deserves to be written and discussed about this.

In this column, however, I'd like to write about the many mathematicians who have already completed the official parts of their careers and have retired from active university duty. As far as research is concerned, retirement is usually not an issue; in fact, at conferences one often doesn't know who among the participants or speakers is or is not retired. This is perhaps as it should be. As mathematicians we are privileged to work in a field that doesn't require labs or large amounts of research funding (although some would be good -- I'm coming back to this later), and it is not uncommon to see active mathematicians who are in their 80s or beyond.

The word 'active', though, can mean different things, and I always find it rather unfortunate when a retired colleague is considered 'no longer active'. Many retired mathematicians, whether or not they engage in published research, contribute a great deal to our departments and our professional organizations. Let me give a few examples relevant to this publication.

By the time you see this article in print or on the screen, it will have been read three times by a long-retired office neighbour and colleague, in three different capacities. I will have asked him, as my friend and mentor, to read this contribution before I submit it to the Editorial Assistant in Ottawa. Then as Co-Editor of the CMS Notes he will have seen it again, and finally, just before it went to the printer, he will have proofread the entire issue in his capacity as Technical Editor of the CMS.

It is true that this colleague of mine, Swami (Srinivasa Swaminathan), has been exceptionally active for two decades into his retirement from official university duties; this has already been recognized in several ways. But he is certainly not the only retired colleague who continues to contribute to our profession in all aspects and at all levels. At this place I should also mention the current Book Reviews Editor for the CMS Notes (Renzo Piccinini) who, just like an earlier Book Reviews Editor, Peter Fillmore, is a past CMS President. The knowledge, experience, and historical perspective these colleagues contributed and continue to contribute, are invaluable -- all this in addition to the time and effort they so freely give to our wider mathematical community.

Another example is the office of Treasurer of the CMS; both the current (David Rodgers) and past Treasurer (Arthur Sherk) have

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Srinivasa Swaminathan, *Dalhousie University*

"Everything of importance has been said before by somebody who did not discover it."

– Adage attributed to Alfred North Whitehead.

Mathematical research is pursued at several universities and other institutions in the world. Mathematicians are producing new results which get published in various journals. Thus it is inevitable that two persons in different parts of the world might come up with similar results on a problem and the date of publication of one of them might precede that of the other. A wishful dream of a mathematician Jane might be that an important contribution by her gets to be known as Jane's Theorem. It may happen that the same result had been published earlier unknown to Jane. Such a situation gives rise to double-named theorems. Thus, in Linear Algebra courses, we have theorems such as Cayley-Hamilton theorem and Gauss-Jordan theorem. There is even triple named one, Cauchy-Bunyakowski-Schwarz Theorem. In Analysis courses we meet with two important theorems: the Heine-Borel Theorem and the Bolzano-Weierstrass Theorem. These two theorems also have interesting stories about their genesis. Such stories can be found in books on the history of mathematics. The Gram–Schmidt process is a method for orthonormalizing a set of vectors in an inner product space, most commonly $\mathbb{R}^n$, named for Jørgen Pedersen Gram and Erhard Schmidt. When Erhard presented the formulae in his paper in *Math. Annalen* 63 (1907) he said that essentially the same formulae were in J.P.Gram's paper in the *Journal fur die reine und angewante Math.* 94 (1883).

It may happen that a major idea may not win universal recognition simply because it got published in a language that is not read or understood by a greater part of the mathematical community. Thus the Norwegian surveyor Kaspar Wessel published the germinal idea of representing a complex number in the x-y plane as early as 1799. The paper was in a Scandinavian language presented to the Royal Danish Academy. The same idea was put forward by the Swiss mathematician Jean-Robert Argand in 1806. The current textbooks refer to it simply as Argand diagram and not as Wessel-Argand diagram.

Quite often it can be that the ultimate source of an idea is not apparent because more able and productive mathematicians, while presenting their own improved versions of germinal ideas, which may not be necessarily their own, do not make explicit references to works of their predecessors. Much of Euler's work on probability is in this category as pointed out by I. Todhunter in his History of Probability. There are also cases where authors are simply unaware that their results had already been found and/or enunciated by others. For example, Cauchy's test for the convergence of a series had been stated by Waring before Cauchy was born, as pointed out by F. Cajori in his book on the history of mathematics.

Questions of priority arise not only in mathematics, as in the classic controversy between Newton and Leibnitz over the discovery of calculus, but also in other subjects, whether scientific or not. Ideas are bound to overlap and results are bound to be repeated. Even Copernicus, who initiated the global way of looking at the heavens, had his predecessors – Herakleides and Aristarchus of ancient Greece, as well as Aryabhatta of India, had suggested that it is the sun rather than the earth that is actually in motion in our galaxy. Thus arose Stigler's law of eponymy, proposed by a University of Chicago professor of statistics, Stephen Stigler: it simply says: 'No scientific discovery is named after its original discoverer.' Stigler names the sociologist Robert K. Merton as the discoverer of this law, consciously making 'Stigler's Law' exemplify Stigler's law!

« Tout ce qui est important a déjà été dit par quelqu'un qui ne l'a pourtant pas découvert. »

Citation attribuée à Alfred North Whitehead.

Il se fait de la recherche mathématique dans bon nombre d'universités et d'autres établissements au monde. Les mathématiciens produisent de nouveaux résultats que publient des revues de toutes sortes. Il est donc inévitable que deux personnes de deux régions du monde présentent une solution semblable à un problème, et que la date de publication de l'une précède l'autre. Il est aussi possible qu'une mathématicienne – appelons-la Marie – rêve de faire une contribution importante qui serait désignée par son nom, le « théorème de Marie » par exemple. Il se peut également que le même résultat ait été publié plus tôt sans que Marie en soit consciente. Ce sont de telles situations qui donnent lieu à des théorèmes à deux noms. Ainsi, dans les cours d'algèbre linéaire, nous avons les théorèmes de Cayley-Hamilton et de Gauss-Jordan. Il existe même un théorème à trois noms, le théorème de Cauchy-Bunyakowski-Schwarz. Dans les cours d'analyse, nous avons deux importants théorèmes : le théorème de Heine-Borel et celui de Bolzano-Weierstrass. Ces deux derniers ont des histoires particulièrement intéressantes, relatées les manuels d'histoire des mathématiques. Le procédé de Gram-Schmidt est pour sa part une méthode utilisée pour construire une base orthonormale à partir d'un ensemble de vecteurs dans un espace muni d'un produit scalaire, connu sous le nom de R^n , en l'honneur de Jørgen Pedersen Gram et d'Erhard Schmidt. Lorsque Erhard a présenté les formules dans un article de *Math. Annalen* 63 (1907), il a mentionné que des formules à peu près similaires se retrouvaient dans un article de J.P.Gram publié dans *Journal fur die reine und angewante Math.* 94 (1883).

Il est possible aussi qu'une idée mineure ne soit pas universellement reconnue simplement parce qu'elle aurait été publiée dans une langue peu commune ou peu comprise par la majeure partie de la communauté mathématique. Ainsi, l'arpenteur norvégien Kaspar Wessel a publié l'idée de représenter un nombre complexe sur le plan x-y dès 1799. L'article, rédigé dans une langue scandinave, a été présenté à l'Académie royale du Danemark. La même idée a été présentée par le mathématicien suisse Jean-Robert Argand en 1806. Or, les manuels actuels parlent simplement du « diagramme d'Argand », et non du « diagramme de Wessel-Argand ».

Bien souvent, la source réelle d'une idée n'est pas apparente parce que des mathématiciens plus talentueux et productifs qui présentent leur propre version améliorée d'une idée originale, qui n'est pas nécessairement la leur, ne mentionnent pas explicitement les travaux de leurs prédécesseurs. Une bonne partie des travaux d'Euler en probabilité entrent dans cette catégorie, comme le souligne I. Todhunter dans *History of Probability*. Il arrive aussi que des auteurs ne soient tout simplement pas conscients que leurs résultats avaient déjà été découverts ou énoncés. Par exemple, le test de convergence d'une série de Cauchy avait été énoncé par Waring avant la naissance de Cauchy, comme le souligne F. Cajori dans son ouvrage sur l'histoire des mathématiques.

Ces questions de priorité ne sont pas propres aux mathématiques. Pensons à la controverse classique entre Newton et Leibnitz à propos de la découverte du calcul différentiel et intégral, et à bien d'autres cas, scientifiques ou non. Il est normal que les idées se recoupent et que les résultats se répètent. Même Copernic, qui a été le premier à observer le ciel de façon globale, avait comme prédécesseurs Heraclides Ponticus, Aristarchus (Grèce ancienne) et Aryabhatta (Inde), qui avaient avancé que le Soleil, et non la Terre, était en mouvement dans notre galaxie. C'est ainsi qu'a vu le jour la loi des éponymes de Stigler, proposée par un professeur de statistique de l'Université de Chicago, Stephen Stigler. Cette loi stipule simplement qu'aucune découverte scientifique ne sera nommée d'après la personne qui est à l'origine de la découverte. Dans l'énoncé de sa loi, Stigler désigne le sociologue Robert K. Merton comme « découvreur » de cette loi, faisant ainsi consciemment de sa loi un exemple de ce qu'il énonçait.

Letters to the Editors

Lettres aux Rédacteurs

The Editors of the NOTES welcome letters in English or French on any subject of mathematical interest but reserve the right to condense them. Those accepted for publication will appear in the language of submission. Readers may reach us at **notes-letters@cms.math.ca** or at the Executive Office.

Les rédacteurs des NOTES acceptent les lettres en français ou anglais portant sur un sujet d'intérêt mathématique, mais ils se réservent le droit de les comprimer. Les lettres acceptées paraîtront dans la langue soumise. Les lecteurs peuvent nous joindre au bureau administratif de la SMC ou à l'addresse suivante : **notes-lettres@smc.math.ca**.

NOTES DE LA SMC

Les Notes de la SMC sont publiés par la Société mathématique du Canada (SMC) six fois l'an (février, mars/avril, juin, septembre, octobre/novembre et décembre).

Note aux auteurs : indiquer la section choisie pour votre article et le faire parvenir au Notes de la SMC à l'adresse postale ou de courriel ci-dessous.

CMS NOTES

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The Editors welcome articles, letters and announcements, which can be sent to the CMS Notes at the address below.

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MARCH 2012

3-4 American Math Society Meeting (Univ. of Hawaii at Manoa, Honolulu)
www.ams.org/amsmtgs/sectional.html

12-16 Classifying fusion categories (Amer. Inst.of Math, Palo Alto, CA)
http://aimath.org/ARCC/workshops/fusioncat.html

17-18 American Math Society Meeting (George Washington Univ. Washington, Dist. Columbia)
www.ams.org/amsmtgs/sectional.html

19-23 Global Arithmetic Dynamics Workshop (Brown University, Providence RI)
http://icerm.brown.edu/sp-s12-w2

25-28 Partial Differential Equations and Applications (Hanoi, Vietnam)
http://www.amath.washington.edu/~kutz/vietnam/

APRIL 2012

14 69th Algebra Day (University of Ottawa)
http://mysite.science.uottawa.ca/neher/algebraday2012.html

16-20 ICERM Workshop: Moduli Spaces associated to Dynamical Systems (Providence, RI)
http://icerm.brown.edu/sp-s12/workshop-3.php

19-22 Fields Institute workshop on Exceptional Algebras and Groups (Toronto)
http://www.fields.utoronto.ca/programs/scientific/11-12/exceptional/index.html

23-25 Conference on Analytical Approaches to Conflict (Royal Military Acad.,Sandhurst, UK)
http://www.ima.org.uk/conferences/conferences_calendar/influence_and_conflict.cfm

23-27 Workshop on p-adic Langlands Program (Fields Institute, Toronto, ON)
http://www.fields.utoronto.ca/programs/scientific/11-12/galoisrep/wksp_p-adic/index.html

MAY 2012

3 Nathan & Beatrice Keyfitz Lectures in Math & Social Sciences (Fields Institute, Toronto, ON)
http://www.fields.utoronto.ca/programs/scientific/keyfitz_lectures/fienberg.html

7-11 From Dynamics to Complexity: conference celebrating work of Mike Shub (Fields Institute, Toronto, ON)
http://www.fields.utoronto.ca/programs/scientific/11-12/dynamics2complexity/index.html

20-27 European Conference on Elliptic and Parabolic Problems (Gaeta, Italy)
http://www.math.uzh.ch/gaeta2012

28-June 3 Theory of Approximation of Functions and Applications (Kamianets-Podilsky, Ukraine)
http://www.imath.kiev.ua/~funct/stepconf2012/en/

JUNE 2012

2-4 CMS Summer Meeting, University of Regina
http://www.cms.math.ca

24-28 Annual Meeting of CAIMS (Fields Institute & U.Toronto, ON)
http://www.fields.utoronto.ca/programs/scientific/11-12/CAIMS_SCMAI/index.html

24-July 6 Séminaire de Mathématiques Supérieures: Graduate School on Probabilistic Combinatorics (CRM, Montréal, QC)
http://www.dms.umontreal.ca/~sms/2012/

JULY 2012

2-6 24th Conference on Operator Theory, (West Univ. Timisoara, Roumania)
http://www.imar.ro/~ot/

8-11 Trends in Set Theory (Stefan Banach International Mathematical Center, Warsaw, Poland)
http://www.impan.pl/~set_theory/Conference2012/

9-15 10th International Conference on Fixed Point Theory and Applications (Cluj-Napoca, Romania)
http://www.cs.ubbcluj.ro/~fptac/

16-20 HPM 2012 History and Pedagogy of Mathematics - The HPM Satellite Meeting of ICME-12 (Daejeon, Korea)
http://www.hpm2012.org

23-27 Algebraic Topology: applications and new directions (Stanford U, Palo Alto, CA)
http://people.maths.ox.ac.uk/tillmann/StanfordSymposium.html

29-August 3 XVIII Brazilian Topology Meeting (XVIII Encontro Brasileiro de Topologia)
www.dm.ufscar.br/~ebt2012/

SEPTEMBER 2012

20-22 Lie and Klein: the Erlangen program and its impact on mathematics and physics (Strasbourg, France)
http://www-irma.u-strasbg.fr/article1173.html

DECEMBER 2012

8-10 CMS Winter Meeting, Montréal, QC
http://www.cms.math.ca

Du bureau du vice-président

Karl Dilcher, *Dalhousie University*

Chapeau aux collègues plus âgés

Ceux et celles d'entre-nous qui travaillons dans une université consacrons beaucoup de temps et d'efforts à enseigner aux étudiants des cycles supérieurs et à les encadrer lorsqu'ils font leur premiers pas vers l'enseignement et la recherche et nous nous intéressons bien souvent activement à leur carrière naissante. Les boursiers postdoctoraux sont aussi très importants pour nos départements, même s'ils ne restent habituellement que quelques années au même endroit. Ils renforcent un département, apportent un nouveau souffle aux efforts de recherche dans l'ensemble et assurent un enseignement à temps partiel, qui est bien souvent en forte demande. On comprend bien l'importance d'offrir le meilleur milieu et les meilleurs systèmes qui soit pour nos jeunes personnes. Ce type d'appui est possible grâce à des programmes à l'intention d'étudiants des cycles supérieurs et à des programmes postdoctoraux offerts par l'ensemble des établissements, de la AARMS au PIMS. Tout cela est indispensable à l'avenir de notre profession. Il faut donc discuter davantage de ce sujet et écrire davantage sur lui aussi.

J'aimerais toutefois traiter dans le présent article des nombreux mathématiciens qui ont déjà terminé le volet officiel de leur carrière et qui se sont retirés de leurs fonctions universitaires. En ce qui concerne la recherche, la retraite n'est habituellement pas un facteur de cause; en fait, au cours des conférences, on ne saurait trop dire qui parmi les participants ou les présentateurs sont retraités ou ne le sont pas. C'est peut-être ainsi que ça devrait être. Comme mathématiciens, nous avons le privilège de travailler dans un domaine qui n'exige aucun laboratoire ni grandes sommes pour le financement de la recherche (bien qu'un certain montant sera bénéfique – je reprendrai ce sujet plus tard). Il n'est pas rare de rencontrer des mathématiciens encore actifs dans les 80 ou plus.

Le mot « actif », toutefois, peut signifier bien des choses. Selon moi, il est dommage de considérer un collègue retraité comme « n'étant plus actif ». De nombreux mathématiciens à la retraite, qu'ils s'intéressent activement à la recherche publiée ou non, contribuent énormément à nos départements et à nos organisations professionnelles. Laissez-moi vous donner quelques exemples pertinents pour la présente publication.

Quand vous aurez parcouru le présent article sur papier ou à l'écran, il aura été lu trois fois par un voisin de bureau et collègue qui est à la retraite depuis belle lurette, et ce, en trois capacités différentes. Je lui ai demandé, comme ami et mentor, de lire cette contribution avant que je ne la présente au rédacteur adjoint à Ottawa. Ensuite, comme corédacteur des Notes de la SMC, il l'aura lu une fois de plus. Finalement, juste avant que l'article soit transmis pour être imprimé, il aura relu toute l'édition en sa capacité de rédacteur technique de la SMC.

Il est vrai que ce collègue à moi, Swami (Srinivasa Swaminathan), a été exceptionnellement actif pendant les 20 années de sa retraite de ses fonctions officielles universitaires; on a déjà reconnu ce fait de bien des façons. Ce n'est certes pas le seul collègue à la retraite qui ne cesse de contribuer à tous les volets et à tous les niveaux de notre profession. À ce point-ci, je devrais peut-être mentionner aussi que le critique littéraire des Notes de la SMC (Renzo Piccinini) qui, tout comme un critique littéraire avant lui, Peter Fillmore, est un ancien président de la SMC. Les connaissances, l'expérience et la perspective historique qu'ont apportées et qu'apportent toujours ces collègues sont très précieux – tout cela s'ajoute au temps et à l'effort qu'ils offrent librement à notre communauté mathématique en général.

Un autre exemple est le bureau du trésorier de la SMC. Le trésorier actuel (David Rodgers) et le trésorier antérieur (Arthur Sherk) ont tout deux apporté à leur poste des dizaines d'années d'expérience dans leurs carrières respectives en enseignement, en recherche et en administration, tant dans le milieu universitaire qu'en dehors de celui-ci. Les deux ont accepté ce poste clé au moment même de prendre leur retraite ou après.

Plus près de chez moi, mon propre département profite toujours de la présence de plusieurs

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CMS Notes
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Secondary Algebra: A quadratic case study

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Editors: John Grant McLoughlin (University of New Brunswick) and Jennifer Hyndman (University of Northern British Columbia)

Jennifer Hyndman recently had a very bright fourth year student ask what doing research would mean for her. Even though we spend our time teaching students how to solve problems, which is one stage of doing research, we often do no teaching about how to create new problems, nor do we let students understand what research is. In this issue we present an article by the previous Editor of this column, Ed Barbeau. He develops the idea that creativity in mathematics can be fostered at the undergraduate level through presentation of material that allows students to formulate questions.

Abstract: Too often, the mathematics curriculum is seen solely in terms of delivering to the student standard topics to be mastered. However, there is a creative side that can be accessed by students still at school; not all new results require years of study of difficult and sophisticated areas. Geometry and combinatorics are two areas where students can enter on the ground floor, but, as I shall indicate by an example, it is possible for a student to obtain an original algebraic result.

While the student in question is particularly strong, I wonder to what extent it is open to students in regular classes to formulate and prove their own results (even if they may be widely known), and how problems might be composed to encourage this to happen.

1. The basic problem. Let me first reconstruct the situation that led to the problem that I posed to students in a correspondence program and an undergraduate competition, and that inspired the original research of one of them. An *oblong number* is any product of two consecutive positive integers. If we examine the sequence $\{2,6,12,20,30,42,56,72,\cdots\}$, we might note that the product of two consecutive oblong numbers is also oblong. For example, 12×20 is equal to the oblong number $240 = 15 \times 16$. Adding 1 to each of the oblong numbers gives a sequence of positive integers of the form $k^2 + k + 1$ namely $\{3,7,13,21,31,43,57,73,91\cdots\}$, with the same property. These empirical observations might be made by an aware student sensitive to patterns. With a little effort, they can be established by deriving the identities:

$$[(x-1)x][x(x+1)] = (x^2-1)x^2$$

and

$$\begin{aligned}[(x-1)^2 + (x-1) + 1][x^2 + x + 1] &= (x^2 - x + 1)(x^2 + x + 1) \\ &= [(x^2+1) - x][(x^2+1) + x] = (x^2+1)^2 - x^2 \\ &= (x^2)^2 + x^2 + 1\end{aligned}$$

Noting that both the forms $x(x+1)$ and $x^2 + x + 1$ are monic quadratic polynomials, we might ask whether these observations can be generalized to numbers of the form $f(x) = x^2 + bx + c$, where b and c are arbitrary integers and x takes consecutive integer values. For example, the product of two consecutive squares is again a square.

One way to approach this problem is to experiment with various examples, and then make an inspired guess as to the value of z generated by the equation $f(x)f(x+1) = f(z)$. Then the proof amounts to just checking algebraically that you are correct.

However, there is another way to approach the problem: transformationally. Suppose that $f(x)$ is an arbitrary monic quadratic polynomial in x. Then $g(t) = f(x+t)$ is a monic quadratic polynomial in t: $g(t) = t^2 + bt + c$. Then, briefly, noting that $g(y) = f(x = y)$ for each y,

$$\begin{aligned}f(x)f(x+1) &= g(0)g(1) = c(1+b+c) \\ &= c^2 + bc + c = g(c) = f(c+x) = f(g(0)+x) \\ &= f(f(x)+x)\ .\end{aligned}$$

Since this sort of manipulation is foreign to most secondary students, let us consider the aspects of the situation that students ought to be made aware of.

First of all, there is change of perspective. The problem is posed as establishing a fact for a particular quadratic and any value of the argument; the realization needs to be made that, by means of a translation of the variable, one can rather prove it for any quadratic and a particular value of the variable, namely 0. Secondly, there is the technical problem of mediating between the two perspectives. Thirdly, it is necessary to make some interpretations: the constant coefficient as the value of a polynomial at 0, and the expression $c(1 + b + c) = c^2 + bc + c$ as the value of $g(c)$. Finally, the evolution of the solution changes the problem. Because one can actually display $f(x)f(x+1)$ as the composition of two quadratics with integer coefficients, the property that we are dealing with f values at integers is subsumed in the more general (and interesting) representation of $f(x)f(x+1)$ as a composite.

2. The general quadratic.

What would the situation be for a quadratic with an arbitrary leading coefficient? Experimentation reveals that $f(x)f(x+1)$ need not be a later value of the quadratic when it is evaluated at integers. However, the work we have done at the end of the last section suggests that we can refocus the problem, to see whether $f(x)f(x+1) = g(h(x))$ for suitable quadratics g and h. This turns out to be true and this result was given as a problem to students at both the secondary and tertiary levels.

Problem. *Let $f(x)$ be a quadratic polynomial. Prove that there exist quadratic polynomials $g(x)$ and $h(x)$ for which*

$$f(x)f(x+1) = g(h(x))$$

Comment. One attempt might be to reduce it to the monic case, an approach that would undoubtedly be difficult for a typical secondary student to consummate but when understood should signify a pretty deep understanding of algebraic relationships. Writing $f(x) = au(x)$, where $u(x)$ is monic, we have that

$$f(x)f(x+1) = a^2u(x)u(x+1) = a^2u(x+u(x))$$

so we can take $g(x) = a^2u(x)$ and $h(x) = x + u(r)$. When $f(x) = ax^2 + bx + c$, this leads to $g(x) = a^2x^2 + abx + ac$ and

$$h(x) = x^2 + \left(1 + \frac{b}{a}\right)x + \frac{c}{a}$$

However, as the following solutions indicate, there are at least three other ways students might tackle this problem, depending on whether they conceive of the quadratic in factored form, in descending powers of x or in terms of completing the square. In the first solution, below, note how it contains the seeds of the generalization we will discuss later. The second solution used the method of undetermined coefficients to obtain a set of five equations in six unknowns. While this may appear formidable, the situation is tractable when the solver realizes that only one solution is needed for an overdetermined system and makes a simplifying assumption. The final solution is an adept completion of the square manipulation. Each of the solutions requires a level of sophistication that we should be encouraging in students planning to go on in science and mathematics.

Solution 1. [A. Remorov] Let $f(x) = a(x - r)(x - s)$. Then

$$\begin{aligned} f(x)f(x+1) &= a^2(x-r)(x-s+1)(x-r+1)(x-s) \\ &= a^2(x^2 + x - rx - sx + rs - r)(x^2 + x - rx - sx + rs - s) \\ &= a^2[(x^2 - (r+s-1)x + rs) - r][(x^2 - (r+s-1)x + rs) - s] \\ &= g(h(x)) \ , \end{aligned}$$

where $g(x) = a^2(x - r)(x - s) = af(x)$ and $h(x) = x^2 - (r + s - 1)x + rs$.

Solution 2. Let $f(x) = ax^2 + bx + c$, $g(x) = px^2 + qx + r$ and $h(x) = ux^2 + vx + w$. Then

$$\begin{aligned} f(x)f(x+1) &= a^2x^4 + 2a(a+b)x^3 + (a^2 + b^2 + 3ab + 2ac)x^2 + (b + 2c)(a + b)x + c(a + b - c) \\ g(h(x)) &= p(ux^2 + vx + w)^2 + q(ux + vx + w) + r \\ &= pu^2x^4 + 2puvx^3 + (2puw + pv^2 + qu)x^2 + (2pvw + qv)x + (pw^2 + qw + r) \ . \end{aligned}$$

Equating coefficients, we find that $pu^2 = a^2$, $puv = a(a + b)$, $2puw + pv^2 + qu = a^2 + b^2 + 3ab + 2ac$, $(b + 2c)(a + b) = (2pw + q)v$ and $c(a + b + c) = pw^2 + qw + r$. We need to find just one solution of this system. Let $p = 1$ and $u = a$. Then $v = a + b$ and $b + 2c = 2pw + q$ from the second and fourth equations. This yields the third equation automatically. Let $q = b$ and $w = c$. Then from the fifth equation, we find that $r = ac$.

Thus, when $f(x) = ax^2 + bx + c$, we can take $g(x) = x^2 + bx + ac$ and $h(x) = ax^2 + (a + b)x + c$.

Continued on next page.

Solution 3. [S. Wang] Suppose that

$$f(x) = a(x+h)^2 + k = a(t-(1/2))^2 + k\ ,$$

where $t = x + h + \frac{1}{2}$. Then $f(x+1) = a(x+1+h)^2 + k = a(t+(1/2))^2 + k$, so that

$$\begin{aligned} f(x)f(x+1) &= a^2(t^2-(1/4))^2 + 2ak(t^2+(1/4)) + k^2 \\ &= a^2t^4 + \left(-\frac{a^2}{2} + 2ak\right)t^2 + \left(\frac{a^2}{16} + \frac{ak}{2} + k^2\right)\ . \end{aligned}$$

Thus, we can achieve the desired representation with $h(x) = t^2 = x^2 + (2h+1)x + \frac{1}{4}$ and $g(x) = a^2x^2 + (\frac{-a^2}{2} + 2ak)x + (\frac{a^2}{16} + \frac{ak}{2} + k^2)$.

3. The generalization.

One student, James Rickards of Greely, ON, raised the situation to a higher level when he realized that he needed to know only that $f(x)f(x+1)$ was a quartic polynomial for which the sum of two of its roots was equal to the sum of the other two. This immediately suggested the generalization that *if the quartic polynomial $f(x)$ has roots r_1, r_2, r_3, r_4 (not necessarily distinct), then $f(x)$ can be expressed in the form $g(h(x))$ for quadratic polynomials $g(x)$ and $h(x)$ if and only if the sum of two of r_1, r_2, r_3, r_4 is equal to the sum of the other two.*

Let us run through the proof of this statement. Without loss of generality, suppose that $r_1 + r_2 = r_3 + r_4$. Let the leading coefficient of $f(x)$ be a. Define $h(x) = (x-r_1)(x-r_2)$ and $g(x) = ax(x-r_3^2 + r_1r_3 + r_2r_3 - r_1r_2)$. Then

$$\begin{aligned} g(h(x)) &= a(x-r_1)(x-r_2)[(x-r_1)(x-r_2) - r_3^2 + r_1r_3 + r_2r_3 - r_1r_2] \\ &= a(x-r_1)(x-r_2)[x^2 - (r_1+r_2)x - r_3^2 + r_1r_3 + r_2r_3] \\ &= a(x-r_1)(x-r_2)[x^2 - (r_3+r_4)x + r_3(r_1+r_2-r_3)] \\ &= a(x-r_1)(x-r_2)[x^2 - (r_3+r_4)x + r_3r_4] \\ &= a(x-r_1)(x-r_2)(x-r_3)(x-r_4) \end{aligned}$$

as required.

Conversely, assume that we are given quadratic polynomials $g(x) = b(x-r_5)(x-r_6)$ and $h(x)$, and that c is the leading coefficient of $h(x)$. Let $f(x) = g(h(x))$.

Suppose that

$$h(x) - r_5 = c(x-r_1)(x-r_2)$$

and that

$$h(x) - r_6 = c(x-r_3)(x-r_4)\ .$$

Then

$$f(x) = g(h(x)) = bc^2(x-r_1)(x-r_2)(x-r_3)(x-r_4)\ .$$

We have that

$$h(x) = c(x-r_1)(x-r_2) + r_5 = cx^2 - c(r_1+r_2)x + cr_1r_2 + r_5$$

and

$$h(x) = c(x-r_3)(x-r_4) + r_6 = cx^2 - c(r_3+r_4)x + cr_3r_4 + r_6\ ,$$

whereupon it follows that $r_1 + r_2 = r_3 + r_4$ and the desired result follows.

Let me allow Rickards to continue in his own words:

I then wondered, how will this continue? What will the condition be for the composition of two third degree polynomials? I tried to construct a proof with only the assumption that the first symmetric polynomials agreed for some division into three groups of three roots of the whole ninth degree polynomial. While writing this out, it became apparent that I lacked something. I then saw that assuming the second symmetric polynomials agreed would be all that I needed. Thus I now had a good idea; I wrote out a proof for a polynomial of degree n^2. The next day or so, I realized that the fact the two polynomials being composed had the same degree was irrelevant; just minor modifications to make this as general as could be, a composition of degrees m and n.

Rickards figured out that the key property of the composite was that its roots could be partitioned into subsets for which all the symmetric polynomials agreed except for the product. This led him to a necessary and sufficient condition for a polynomial of degree mn to be the composite of polynomials of degrees m and n. He then addressed the determination of the composition factors of these degrees when the composite was given. He noted that while one could not generally know the actual roots of the polynomial, the coefficients of the composite factors depended only only on knowing the value of the (equal) symmetric functions of roots in each of the partitioning sets, and these values could be retrieved from the coefficients of the given polynomial. It is convenient to give the proof for a monic polynomial, and then derive the general case; the details are found in [2].

The relating of the monic to the general situation is a nice exercise for students. Suppose that $f(x)$ is a polynomial of degree nm and leading coefficient a, so that $f(x) = au(x)$ for some monic polynomial $u(x)$. Then we show that $f(x)$ is a composite of polynomials of degrees m and n if and only if $u(x)$ is so. Suppose that $f(x) = g(h(x))$ where $g(x)$ is of degree m with leading coefficient b and $h(x)$ is of degree n with leading coefficient c. Then, by comparison of leading coefficients, we have that $a = bc^m$. It can be checked that $u(x) = v(w(x))$ where $v(x) = (bc^m)^{-1}g(cx)$ and $w(x) = c^{-1}h(x)$.

On the other hand, suppose that $u(x) = v(w(x))$ for some monic polynomials $v(x)$ and $w(x)$ of respective degrees m and n. Then $f(x) = g(h(x))$ with $g(x) = au(x)$ and $h(x) = v(x)$.

4. Is Rickards' result new?

I was enchanted by the elegance of Rickards' result. While the determination of a different criterion for a quartic to be the composite of two quadratics actually appears in my book [**1**] (problem 1.9.8, pages 44, 266), it is from the more pedestrian standpoint of a condition on the coefficients. Specifically, $ax^4+bx^3+cx^2+dx+e$ is a composite of two quadratics if and only if $4abc-8a^2d = b^3$. I was completely unaware of this new result, and a check of colleagues, the literature and the internet did not reveal that it was previously known.

Whether it is actually new is open to question. While the composition of polynomials does not appear to have received much attention, it is conceivable that over the past three hundred years, someone might have addressed the issue. However, such a result, if published, could have appeared in an obscure place and be impossible to track down. It seemed pretty enough to warrant appearing in a widely circulated current journal, regardless of its status.

5. Conclusion. It seems clear that, if a curriculum is to be successful in preparing mathematics students for later study, it has to go beyond a straight presentation of results. Students require material that engages them, so that they acquire facility with the conventions and distinctions of mathematics and are able to make judgments about how a situation might be approached. Therefore, we need to be on the lookout for investigations and problems that encourage different perspectives and the search for connections.

In this note, I have presented one situation and mentioned issues that might arise. I hope that teachers may be able to present other examples, and that eventually exercises and problems that might lead to open-ended investigations by students might be more prominent in textbooks. As educators, we need to develop other case studies and then encourage teachers to try them out in their own settings. I have not had the opportunity to attempt this in a regular classroom situation. Its evolution is probably highly dependent on the context; it may happen that the discussion goes in a completely different direction and other questions emerge.

There are important issues pertinent to the preparation of students bound for science, technology and mathematics. Should such students be able to negotiate the subleties of algebra usage illustrated by this example? If so, what are the implications for teacher training, the syllabus, the classroom experience and examinations? What preparation should be occurring all through the algebra sequence so that students attain both the perspective and the skills to manage it? What is the appropriate balance between presentation of such material by the teacher and investigation by individual students and groups? I invite teachers to try this example with their own classes and circles.

References

1. E. J. Barbeau, *Polynomials.* Problems Books in Mathematics, Springer, New York, 2003. ISBN 0-387-40627-1
2. James Rickards, *When is a Polynomial a Composition of Other Polynomials? American Mathematical Monthly* **118:4** (April, 2011), 358-363.

In praise of older colleagues, continued from cover

brought to their positions decades of experience in their respective careers in teaching, research, and administration, both within and outside of academia. Both took on this key position into (or after) retirement.

Closer to home, my own department continues to benefit from the presence of several other retired colleagues. Last September, when a younger faculty member had to go on sick leave, two retired colleagues stepped in on very short notice and taught, for almost the whole term, the two courses that were affected. Given the increasing academic staffing shortages, two math camps held in our department depend to a large extent on the help (and in fact leadership) of some of our retirees. Most likely the same will be the case elsewhere.

It is therefore in the universities' and the departments' best interest if we as still (officially) active faculty members and as heads, chairs, or directors, extend the same kind of welcome and provide similar support to our retired colleagues as we do to postdocs. The status of Adjunct Professor or, where appropriate, Emeritus or Emerita, is usually available to retired faculty, and it normally comes with privileges related to supervising or co-supervising graduate students, or serving on thesis committees. While office space is usually at a premium, some shared space can often be found. Library privileges and computer accounts should be a given, as should be some reasonable secretarial support.

Research support for our retired colleagues has unfortunately deteriorated in recent years. With NSERC's increased and exaggerated emphasis on 'HQP' (an awful term, by the way), it has now become very difficult for even the more prominent retirees to retain their grants. This is particularly unfortunate since retired colleagues will no longer have access to Faculty or University travel funds. Of course the point has been made before in this publication and elsewhere that even a small grant can make a big difference to a mathematician.

Sometimes retired colleagues move away to be closer to family, or for other reasons, including research opportunities. But this is often a two-way street: others move to our communities, and we should welcome them into our midst; we can only win. I do hope this doesn't sound patronizing, especially the use of 'them' and 'us'; it is not meant to be -- the point I want to make is that we are all one family.

Just as large multi-generational families often work best (I grew up in one myself in rural Germany), having all generations of mathematicians under one department roof will be of great benefit to all. In fact, I believe that the young and middle-aged benefit from the wisdom and service of the old more than the other way around. Let's try to equalize this a little bit.

Chapeau aux collègues plus âgés, suite de la page 5

autres collègues à la retraite. Au mois de septembre dernier, lorsqu'un plus jeune membre du corps enseignant a dû prendre un congé de maladie, deux collègues à la retraite sont venus le remplacer sans beaucoup d'avis et ont donné, pendant tout le trimestre pratiquement, les deux cours qui étaient touchés. Vu la pénurie de personnel universitaire, deux camps mathématiques organisés dans notre département sont tributaires en grande partie de l'aide (et en fait du leadership) de quelques-uns de nos retraités. Il en sera ainsi ailleurs probablement.

Les universités et les départements ont donc intérêt à ce que nous, en tant que membres enseignants toujours (officiellement) actifs et en tant que chefs, présidents ou directeurs, appuyons de manière semblable et accueillons aussi ouvertement nos collègues à la retraite que les étudiants postdoctoraux. Le statut de professeur auxiliaire ou, le cas échéant, de professeur émérite peut habituellement être accordé à des membres enseignants à la retraite. On y associe habituellement des privilèges liés à la supervision ou à la supervision conjointe des étudiants des cycles supérieurs ou à la possibilité de siéger aux comités de thèses. Bien que les locaux à bureaux soient très en demande, on peut souvent trouver des installations à partager. On devrait naturellement offrir des privilèges de bibliothèque et des comptes informatiques, de même qu'un certain appui de secrétariat raisonnable.

L'appui à la recherche pour nos collègues à la retraite a malheureusement baissé ces dernières années. Puisque le CRSNG met de plus en plus l'accent – et de manière exagérée – sur le PHQ (terme très laid, soi dit en passant), même les personnes retraitées les plus illustres ont beaucoup de difficulté à conserver leurs subventions. Cette situation est particulièrement déplorable parce que les collègues à la retraite n'auront plus accès aux fonds de voyage des professeurs ou des universités. On a déjà indiqué dans cette publication et ailleurs que même une petite subvention peut avoir de grandes conséquences pour un mathématicien.

Les collègues à la retraite déménagent parfois pour se rapprocher de leur famille ou pour d'autres raisons, y compris des occasions de recherche. C'est bien souvent un exercice à deux sens : d'autres personnes arrivent dans nos collectivités, et nous devrions les accueillir dans notre milieu; seuls des bienfaits ne peuvent en découler. J'espère que je ne donnerai pas d'impression condescendante en employant des termes comme « eux » et « nous ». Ce n'est pas ce que je visais – ce que je voulais faire valoir, c'est que nous formons une seule famille.

À l'instar des grandes familles multi-générationnelles (je suis issu d'une telle famille provenant d'une région rurale d'Allemagne), regrouper toutes les générations de mathématiciens sous un même toit au sein du département sera profitable à tous. En fait, je crois que les jeunes et les personnes d'âge moyen profitent davantage de la sagesse et du service des personnes plus âgées que dans le sens contraire. Essayons d'égaliser tout cela un peu.

The Sun Life Financial Canadian Open Mathematics Challenge

On November 2, 2011, thousands of high school students from all across Canada participated in the 2011 Sun Life Financial Canadian Open Mathematics Challenge (COMC). The COMC is Canada's national mathematics competition for any student with an interest in mathematics, and encourages students to explore, discover, and learn more about mathematics and problem solving. The COMC is presented by the CMS in partnership with the University of Toronto and Université Laval, with assistance from Carleton University and the University of Ottawa.

This year's COMC consisted of 12 problems, in 3 sets of increasing difficulty. The following problem was considered by the problems committee to be the most difficult problem in the contest:

Let $f(x) = x^2 - ax + b$, where a and b are positive integers.

(a) Suppose $a = 2$ and $b = 2$. Determine the set of real roots of $f(x) - x$, and the set of real roots of $f(f(x)) - x$.

(b) Determine the number of pairs of positive integers (a, b) with $1 \leq a, b \leq 2011$ for which every root of $f(f(x)) - x$ is an integer.

Prize Draw

Adam Albogatchiev was the winner of the 2011 COMC Prize Draw, receiving a $1000 prize from the CMS. Congratulations Adam! The CMS also distributed t-shirt prizes to over 400 students.

The award winning top ranked students in Canada for the 2011 COMC are:

Matthew Brennan

Upper Canada College (Toronto, ON) - National Gold Medal

James Rickards

Colonel By Secondary School (Ottawa, ON) - National Silver Medal

Steven Yu

Pinetree Secondary School (Coquitlam, BC) - National Gold Medal

Kevin Zhou

Woburn Collegiate Institute (Scarborough, ON) - National Bronze Medal

In addition, the CMS introduced an expanded awards program for the 2011 competition that serves to better recognize student performance at the provincial, regional, and grade levels. Full award listings are available at: **http://cms.math.ca/Competitions/COMC/2011/results.html**

Based on their COMC scores, 66 students have received a direct invitation to write the Sun Life Financial Canadian Mathematical Olympiad (CMO), Canada's national advanced mathematics competition. 14 more students have been invited to write the CMO based on their results in the Sun Life Financial Repêchage Competition, which acts as a qualifier for the CMO.

BRIEF BOOK REVIEWS

by Srinivasa Swaminathan, Dalhousie University

Affine Algebraic Geometry
– The Russell Festschrift

Edited by Daniel Daigle, Richard Ganong & Marius Koras

CRM Proceedings and Lecture Notes: 54

ISBN 978-0-8218-7283-3, xvii + 334pp, AMS 2011

Affine geometry is geometry not involving any notions of origin, length or angle. Affine space is an abstract structure that generalizes the affine geometric properties of Euclidean space. Affine algebraic geometry is the study of affine spaces and algebraic varieties closely resembling them. The present volume contains the proceedings of an international conference held in June 2009 on the occasion of the 70th birthday of Professor Peter Russell, Director of CRM (2009-2011), upon his retirement from McGill University. There are 19 papers all in the area of affine algebraic geometry dealing with the following subjects: automorphisms and group actions, surfaces, embeddings of rational curves in the affine plane, and problems in positive characteristic geometry. These are also some of the themes running through substantial body of work done by Peter in the subject. A foreword by Peter shares some personal reminiscences on the development of the subject.

Perspectives on Noncommutative Geometry

Edited by Masoud Khalkhali & Guoliang Yu

Fields Institute Communications 61

ISBN 978-0-8218-4849-4, viii + 163pp, MAS 2011

Noncommutative geometry is the study of noncommutative spaces, i.e., spaces represented by a noncommutative algebra that replaces the coordinate algebra of commutative spaces. Examples include highly singular spaces such as the space of leaves of a foliation, the unitary dual of a noncompact group, and more generally, 'bad quotients' of classical spaces. Initiated and pioneered by Alain Connes since 1980, the subject was inspired by global analysis, operator algebras, and quantum physics, as these show up in areas such as index theory, foliation theory and quantum statistical mechanics. Its main applications were to settle some conjectures such as the Novikov conjecture and the Baum-Connes conjecture in topology and analysis, using tools like cyclic cohomology, K-theory, K-homology and K–K theory. After some further development, strong interaction arose between number theory, algebraic geometry, theory of motives and quantum field theory on noncommutative geometry. The papers in this volume are based upon lectures by Alain Connes and the invited speakers at the workshop. The book will be useful to graduate students and researchers in both mathematics and mathematical physics.

Differential Geometry of Curves and Surfaces

By Thomas Banchoff and Stephen Lovett

A.K. Peters, Ltd. 2011, xvi + 331 pp, 2010

ISBN 978-1-56881-456-8

Differential Geometry of Manifolds

By Stephen Lovett

A.K. Peters, Ltd. 2011, xiii + 421 pp, 2010

ISBN 978-1-56881-457-5

Differential geometry studies properties of, and analysis on, curves, surfaces and higher dimensional spaces using tools from calculus and linear algebra. The types of questions in differential geometry fall into two categories: local properties, i.e., properties of a curve or surface defined on a neighborhood of a point; and global properties referring to properties of curves and surfaces taken as a whole. The first book deals with the classical theory in spaces of one or two dimensions using only vector calculus and linear algebra as prerequisites. The second book continues the development of the subject by studying manifolds which form a natural generalization of regular curves and surfaces to higher dimensions.

The first book can be used as a text for a one-semester undergraduate course. Interactive computer applets are provided for the book to be used in computer labs, in-class illustrations, exploratory exercises and also as intuitive aides for the reader. Each section concludes with exercises. Chapters 1 through 4 cover: fundamental notions of curvature, torsion, evolutes, osculating circles and spheres; global properties such as closedness, concavity, winding numbers and knottedness. Chapter 5 is on regular surfaces, tangent planes and orientability. Chapter 6 deals with three-dimensional geometry focusing on metric tensor, Gauss maps, Gaussian curvature and mean curvatures. Tensors are treated in Chapter 7 establishing the famous Theorema Egregium, the celebrated classical result on Gaussian curvature depending on metric tensor only. In Chapter 8, geodesics and the famous Gauss-Bonnet theorem are discussed.

The second book does not rely on the first one and can be read independently. It begins with a chapter on functions from an n-dimensional spaces to an m-dimensional one. Chapter 2 discusses moving frames. Chapter 3 is on the category of differentiable manifolds. Chapter 4 develops analysis on differentiable manifolds. Chapter 5 introduces Riemannian geometry. Chapter 6 is devoted to applications of manifolds to various topics in physics. Exercises are provided at the conclusion of each chapter. Three appendices deal with some topology and linear algebra requirements. The book gives a concrete introduction to the theory of manifolds at an advanced undergraduate or beginning graduate level.

Convexity: An Analytic Viewpoint

Barry Simon

Cambridge Tracts in Mathematics, 187

Cambridge University Press, 2011

ISBN 978-1-107-00731-4

Reviewed by Vicente Montesinos, Universidad Politécnica de Valencia

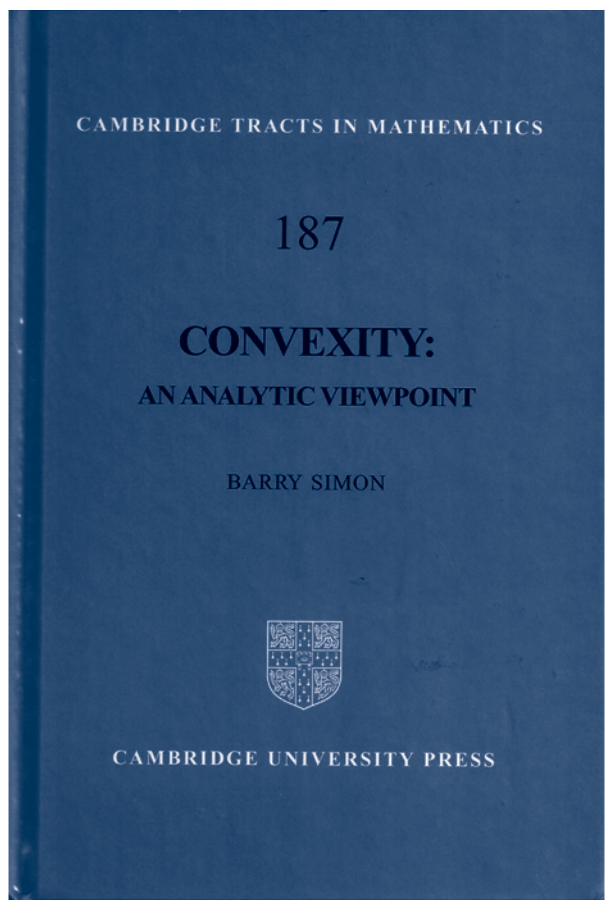
Convexity is an area of Mathematics that permeates many others, from Geometry (finite and infinite-dimensional) and Analysis (mainly inequalities[1], but also differentiability, measure theory, optimization, interpolation, etc.), to Statistics, Probability Theory, Numerical Calculus, and many others.

This book is an excellent presentation of the fascinating field of convexity theory and its use in inequalities. It is written with a great care in developing the subject from the basic concepts to the most advanced topics in a uniform and measured way, without avoiding explanations, examples, remarks and comments that turn the process of learning into a lively, suggestive and documented experience.

Chapter 1, "Convex functions and sets", lays down the basics, starting from the very definition of a convex function, going through to the concept of Legendre transform, and setting the so-called "Young's inequality". In between, the reader can find out the notion of a convex set, when midpoint convexity implies convexity, the classical necessary and sufficient condition for convexity of a real-valued function on an open interval in $\mathbb{R}$ (on an open convex subset of $\mathbb{R}^n$) in terms of the second derivative (respectively, of the Hessian), his/her first contact with the gauge of a convex set containing Δ, with convex cones, tangents to convex functions, and pseudoconvex sets. Here, and in the rest of the book, the author presents results in the finite-dimensional case — sometimes as a motivation, sometimes because of their intrinsic importance — to later focus on the infinite-dimensional picture.

Chapter 2 is fully devoted to Orlicz spaces and, together with the short introduction in Chapter 1, may be successfully used as a quick course on this subject. The presentation is clear, goes to the point, and the author includes many examples in order to demonstrate the role that elements like Young functions or the Δ_2 condition play in it. Duality theory is discussed at length — the Legendre transform plays a crucial role here — culminating with the description of the dual space of an Orlicz space satisfying the Δ_2 condition. Along the way, another proof (the fourth one!) of Hölder's inequality — this time based on Young's inequality — is given.

Chapters 3, 4, and 5 form an excellent introduction to locally convex spaces and duality theory. It can be successfully used as a crash course on these topics, with an emphasis on convexity and weak topologies. Locally convex spaces, i.e., topological vector spaces having a basis of convex neighborhoods of the origin, naturally play a central role. In this context, the basic results on convexity that blend topology and linear algebra are seen in its natural abstract setting, emphasizing what is intrinsic to the nature of convexity regardless of dimension. Basic concepts are introduced in Chapter 3: topological vector spaces, boundedness, completeness, dual pairs and the weak topology, gauges and seminorms, normability, locally convex spaces, metrizability, etc. A selected set of examples (finite-dimensional spaces, L^p for $0 < p < 1$, H^p for $0 < p < 1$, the Schwartz test for function spaces, the space of distributions and spaces of holomorphic mappings) is discussed for the better understanding of the reader.



The brief Chapter 4 presents typical separation theorems. Its brevity is a consequence of the fact that the fundamental Hahn–Banach theorem is already discussed in Chapter 1.

Chapter 5 focuses on duality theory. After describing the closed convex hull of a set and showing how weak convergence implies convergence of convex combinations (Mazur's theorem), the bipolar and the Alaoglu —Bourbaki theorems — the basic tool providing compactness "for free" — are established, and its implications for the discussion of gauges and polar sets explored. The goal of the author is to discuss convexity for functions and sets (mostly for the former and as a subsidiary

1 References - Bibliographic item [DL] is now complete. Please update to: Proc. Amer. Math Soc. 140 (2012), 1151-1157.

tool for the latter). Taking advantage of the number of results already presented, he finds it convenient at this stage to extend the previous study on the Legendre transform to functions with values in $\mathbb{R} \cup \{\infty\}$ with the sole requirement of being bounded below by an affine function. Fenchel's Theorem on double Legendre transform appears as a result on convex envelopes. The chapter ends by briefly discussing the Mackey topology (proving, for example, the Mackey–Arens Theorem), the strong topology, and the concept of reflexive space (proving, in particular, that every Montel space is such).

Chapter 6 has the title "Monotone and convex matrix functions". It extends convexity of real-valued functions to functions from self-adjoint operators to themselves that are generated via the functional calculus. The main result presented here (its hard part proved in the next chapter, although a second proof is carried out in Chapter 9) is Loewner's theorem characterizing real-valued functions on $(-1,1)$ (and then on all (a,b)) that lie in $M_\infty(-1,1)$, i.e., when acting on the class of n x n self-adjoint matrices, behave monotonically on those having their spectrum in $(-1,1)$ (for all $n \in \mathbb{N}$). Many of the preliminaries that are needed for the hard part of the proof are established here. Not only this, but very interesting consequences of the integral representation that Loewner's theorem ensures are discussed here, as the connection between functions in $M_\infty(a,b)$ and functions having analytic continuation to suitable subsets of $\mathbb{C}$. Tools for the proof of Loewner's theorem are divided differences and Loewner and extended Loewner matrices. Extension to open subsets of $\mathbb{R}^n$ are also considered. The chapter ends with an analysis of matrix convex functions, needed for the second proof of the theorem.

Chapter 7 presents the Bendat–Sherman proof of Loewner's theorem. This is the first proof offered (there is a second one in Chapter 9) in the book. It relies on two theorem of independent interest: the Bernstein–Boas theorem on C^∞-functions on $(0,1)$ with positive even derivatives, and the Hausdorff Moment theorem.

Chapter 8 is the first of a series of chapters explaining the fundamental role that extreme points of convex sets play in questions of structure, approximation and optimization. It makes a brief and enlightening tour around the basic results on the subject — the Minkowski–Carathéodory Theorem and the Krein–Milman Theorem. The presentation follows the usual approach. What is really remarkable is the series of worked examples that help to understand the delicate points of the theory and provide an already substantial insight into what the reader should expect for the rest of the book. For example, besides some expected finite-dimensional examples, including the n-simplex, the author discusses the cases of the closed unit ball of $C[0,1]$, the closed unit ball of the dual space of $C[K]$, how ergodic measures appear as extreme points of certain subsets of this dual space, or irreducible representations as extreme points of a certain subset of $C(G)$, G a locally compact group. The chapter ends by using the machinery of extreme points in proving the Stone–Weierstrass Theorem (following de Branges) and the Lyapunov's Theorem for nonatomic finite measures.

Chapter 9, the second on the subject of extreme points, has the title "The Strong Krein–Milman theorem". By this, the author understands the result that any point of a compact convex subset of a locally convex space is the barycenter of a measure supported by the closure of the set of its extreme points. Bauer's theorem (and Milman's "converse" to the Krein–Milman theorem as a consequence) precedes a very enlightening discussion on the limits — and yet the broad scope of the applicability — of the theorem in the title. For the limits, a selection of clarifying examples (uniformly convex spaces, Lipschitz functions, the Poulsen simplex, the moment problem,...) is presented. Then three important results are proven with the use of the Strong Krein–Milman theorem: Bernstein's theorem on completely monotone functions — both for bounded and unbounded functions — , Bochner's on positive definite functions, and Loewner's on functions monotone on matrices. While the proofs of the two first results are more or less in the literature, the proof of the last one introduces some changes with respect to the Hansen — Pedersen approach.

Chapters 10 and 11 are on Choquet theory. The first one deals with the question of existence of the representing measure, the second with uniqueness. The author concentrates on the metrizable (i.e., the separable) case, arguing — and we completely agree — that in "practical applications" this is what matters. The Choquet ordering is introduced to prove the Choquet theorem on a metrizable compact convex subset of a locally convex space, and maximal measures are identified as those supported on the set of extreme points. The second one introduces the concept of Choquet simplex, and concentrates on the proof of the Choquet–Meyer Theorem relating uniqueness of representing measures and simplices. There is no need, in this case, of demanding separability, although the proof, as it is, assumes this condition. The author warns the reader that, to properly complete the proof in the general case, some issues regarding Baire versus Borel sets need to be adjusted. Equipped with this result, the author discusses previous examples in order to see whether the involved convex compact sets are, or not, simplices. Particularly interesting is the revisitation of the Poulsen simplex.

Chapter 12 is the first of a series of chapters — forming the last part of the book — with an emphasis on inequalities. It is the only one where analytic functions and inequalities related to them are the central issue. Here, the reader will find Hadamard's Three-Circle and Hadamard's Three-Line theorems, as well as some applications: the Stein and the Riesz–Thorin Interpolation theorems, and the Young, Generalized Young, Sobolev and Strichartz inequalities.

Chapter 13 focuses on the Brunn–Minkowski inequality and log concave functions. The discussion around the isoperimetric inequality is a fascinating one. This inequality is obtained as an application of the Brunn–Minkowski inequality. In turn, the particular case of this last inequality for two open convex subsets is proven by using Prékopa's theorem on log concave functions. At the end, the general case of arbitrary (Borel) sets is proven by an approximation process.

Chapters 14 and 15 have a common title: Rearrangement inequalities (I and II, respectively). The first one deals with the general subject and concentrates on the Brascamp–Lieb–Luttinger (BLL, in short) inequalities. Finite sequences are a good motivation for what follows: symmetric rearrangements of functions, the Riesz's Rearrangement Inequality, and the Hardy–Littlewood–Pólya theorem (announced here but proved in the next chapter). The main result in this chapter is the proof of the BLL inequalities, first in the 1-dimensional case, then in n dimensions. Several applications are presented, including some general isoperimetric inequalities, some related to potentials, to the Dirichlet ground state energy, to torsional rigidity, and to Coulomb energy.

As mentioned, Chapter 15 focuses on the Hardy–Littlewood–Pólya theorem, presenting different — regarding on their generality — versions, with increasing levels of difficulty, the simplest one being that if a and c are (finite) sequences and a^*, c^*, denote, respectively, their decreasing rearrangements, and $\sum_{j=1}^{k} c_j^* \leq \sum_{j=1}^{k} a_j^*$ for k= 1,2,…,n, then $\sum_{j=1}^{n} \varphi(c_j) \leq \sum_{j=1}^{n} \varphi(a_j)$ for any convex function φ. The author develops a comprehensive program. He starts with the case where equality for $k = n$ is demanded, then goes to an analog for positive matrices; after that, the equality condition above is dropped, and then he tackles the version for the absolute values of two sequences. The infinite-dimensional case is done for discrete variables, finalizing by taking in consideration the case of general measure spaces. Several interesting applications (for example, Hadamard's or Minkowski's Determinantal inequalities) end the chapter.

Chapter 16, the last of the body text, discusses a final convexity inequality that connects convexity and entropy, important in statistical mechanics, information theory and the spectral theory of Jacobi matrices.

Chapter 17 deserves special consideration. An excellent decision on the part of the author has been to collect notes — including historical notes — , extensions and remarks on the material in a single chapter at the end of the book. In this way, the main course of events is not disturbed. Here the reader will find a truly amazing amount of complementary information that, in 34 pages, gives an overall picture of the subject covering issues from the the past to the most recent advances in the area. The author works extensively on attributions, discussing issues of priority, precedents, collateral developments and related results; all of them are properly documented and referred to. He actually proves in full some extensions of results in the text, like Muirhead's theorem or an analog of Bernstein's theorem, this time for C^∞ functions on $(0,1)$ having all derivatives positive. Of course, most of the results are presented without proofs; however, precise references and even sketches of proofs are given (as in the case of Boas' generalization of the the Bernstein–Boas theorem, or of Carathéodory's theorem on convex combinations in finite-dimensional spaces). It is impossible here to illustrate to the reader of this report the richness of the material presented. This reviewer found all the historical remarks and the complementary notes on the basics of Functional Analysis and of Banach space theory in particular very appropriate, and read with great interest the many others on convexity and inequalities.

Definitions of new objects are also found here. For example, the reader will face several notions not discussed in the book (uniform convexity, points at infinity, convex programming, Helly's theorem, subharmonicity, convex inequalities for matrices. and minimax principles, among others).

The book does not contain a list of exercises. At first glance this can be seen as an impediment — specially because the format in which the book is written makes it a good choice for a textbook in (advanced) convexity. It turns out, however, that the impressive amount of worked examples, plus the large collection of notes, remarks, suggestions for further reading, alternative proofs, variations and complements on the subject matter, is a good substitute, and even exceed the usefulness, of a possible list of exercises.

The Subject index contains, certainly, the list of the most important concepts and results in the text. Maybe a longer, more detailed index — and a list of symbols — would have been helpful in the sense of quickly directing the reader through the huge amount of information disseminated in the book. On the other hand, the Reference list is large and well selected, and even an Author's index is included.

Remarkably, there are very few typos, and the overall aspect — typography, the inclusion of some suggestive figures and the quality of this well-crafted edition — makes reading and consulting this book a most pleasant experience.

Summarizing, and somehow repeating what has already been said, this is a splendid book on convexity theory with a strong emphasis on analytic tools, going from the basic results to advanced topics. It is very well written, extremely readable, full of examples, and contains a huge amount of information, sometimes in the form of notes, remarks, historical references and suggestions for further reading. A perfect choice as a graduate text, advanced seminars and courses, and an invaluable source of information for the researcher.

On the dynamics of black hole formation

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One of the most remarkable predictions of general relativity concerns the behaviour of massive stars. In his celebrated work, which dates back to 1930 and earned him a Nobel Prize in 1983, Chandrasekhar showed that a relativistic treatment of the equation of state for degenerate matter implies that, after having burnt up its nuclear fuel, a sufficiently massive star will collapse gravitationally and eventually settle into a configuration now called a *black hole* [5]. The space-time geometry near a black hole has a very special structure, which is reflected through the presence of an *event horizon* acting as a natural boundary for the outer region of space-time and a *singularity* in the geometric fabric of space-time. The *cosmic censorship* conjecture of Penrose asserts that any such singularity should be hidden to an external observer by the presence of an event horizon.

Understanding the dynamics of black hole formation in general relativity is a problem of formidable mathematical difficulty because of the non-linear nature of Einstein's field equations of gravitation and the fact that neither the space-time geometry nor its topology are pre-set when solving these equations. Indeed, gravitation differs from all other field theories in that the geometric background that carries the gravitational field is determined by solving the field equations.

Let's start with some basic facts. Space-time is a 4-dimensional manifold, M_4, endowed with a metric $g = (g_{ij})$ of Lorentzian signature $(+, -, -, -)$. The tangent space at each point of space-time is thus isomorphic to the Minkowski space. The world lines of massive test particles in M_4 correspond to time-like geodesic curves, and the light rays propagate along null geodesics. The *Jacobi equation* governs the behaviour of pencils of neighbouring geodesics through the curvature tensor, i.e. if the *Jacobi field* n connects two such geodesics, then

$$\frac{d^2 n^i}{d\lambda^2} R^i_{uju} n^i = 0$$

The gravitational field and the space-time curvature are therefore closely related in any relativistic theory of gravitation. The condition to obtain Newtonian gravity in the classical limit leads to *Einstein's field equations of gravitation*,

$$R_{ij} - \frac{1}{2} R_{gij} = 8\pi T_{ij} \tag{1}$$

The *Einstein tensor* R_{ij}, a symmetric tensor field that depends on the first and second derivatives of the metric, is divergence-free as a consequence of the Bianchi identities. The *energy-momentum tensor* T_{ij} of the matter fields interacting with gravity is symmetric and divergence-free due to the field equations for the matter fields. In the absence of sources, the Einstein field equations reduce to $R_{ij} = 0$.

Rigorous mathematical progress on the understanding of black-hole formation has been slow, but recent major work due to Christodoulou, [4], has essentially settled the question, as we will further explain. Let's begin with the celebrated Penrose singularity theorem, [5], which gives necessary conditions for the formation of a singularity.

Theorem 1 *A space-time (M_4,g) cannot be null geodesically complete if the following conditions are satisfied:*

(i) $R_{ij}k^ik^j \geq 0$ for all null vectors k;

(ii) there is a non-compact Cauchy surface in (M_4,g);

(iii) there is a closed trapped surface in (M_4,g), that is a space-like two-surface T such that the two families of null geodesics orthogonal to T are converging at T.

It should be noted that a similar result was proved by Hawking, where condition (i) is replaced by a stronger energy condition and the conditions (ii) and (iii) are weakened, [5]. We also note that Hawking showed that Penrose's argument can be turned upside down to apply to the time-reversed situation, on a cosmological scale, giving rise to past singularities. We will not discuss these cosmological applications here, [5]. Penrose's theorem does not give any details on the dynamical aspects of the formation of singularities and event horizons in terms of the Cauchy data. From a classical theorem of G.D. Birkhoff, [5], we know that all the spherically symmetric solutions of the *vacuum* Einstein field equations, $R_{ij} = 0$, are necessarily *static* and given by the one-parameter family of Schwarzschild metrics,

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 = \left(1 - \frac{2M}{r}\right) dr^2 - r^2(d\theta + \sin^2\theta\, d\ '^2),$$

which describe the external gravitational field of a spherically symmetric black hole in equilibrium ($r = 2M$ is the event horizon and the singularity is located at $r = 0$). It follows that any study of the dynamics of *spherical* gravitational collapse and singularity formation will require the coupling of the gravitational field to some matter field. In an earlier crucial result, [1], Christodoulou addressed this question in the case of gravitation coupled to a massless scalar field in spherical symmetry. To state this result, we will use advanced null coordinates $(u, r, \theta,\ ')$ in which every spherically symmetric space-time metric can be written in the form

$$ds^2 = e^{-2v} du^2 - 2e^{v+\lambda}\, du\, dr - r^2\, (d\theta^2 + \sin^2\theta\, d\ '^2), \tag{2}$$

where $\lambda = \lambda(u,r)$, $v(u,r)$. The hypersurfaces $u = constant$ are future-pointing null geodesic cones, and the cross-sections $r = constant$ of these cones are diffeomorphic to 2-spheres.

We then consider the Einstein field equations describing the interaction of a gravitational field with a massless scalar field,

$$R_{ij} - \frac{1}{2} R g_{ij} = 8\pi (\partial_i \phi \partial_j \phi - \frac{1}{2} g_{ij} \partial_k \phi \partial_k \phi), \quad g^{ij} \mathrm{r}_i \mathrm{r}_j \phi = 0,$$

in the spherically symmetric geometry of (2). The Cauchy data for this system can be expressed in terms of a single scalar mass function $m(r)$ on the future-pointing null geodesic cone C_u^+ given by $u = 0$ as follows.

Theorem 2 *Let* $r_2 > r_1 > 0$ *be such that* $0 < \delta : r_2/r_1 - 1 < 1/2$, *and suppose that* $2[m(r^2) - m(r_1)] \geq \eta(\delta)|(1+\delta)^{-1}(r_2 - r_1)$, *where* $\eta(\delta) := \log\frac{1}{2\delta} + 5 - \delta$. *Then*

(i) a trapped region forms in the future and ends at a strictly space-like singular boundary B*;*

(ii) near B*, we have* $R_{ijkl}R^{ijkl} \geq c/r^6$.

In subsequent work, [2], [3], Christodoulou showed that singularities not hidden by an event horizon can occur when starting with different Cauchy data, but that these singularities are not stable, i.e. the data leading to this type of behaviour are of co-dimension 2 in the space of initial data of bounded variation. This result proves the validity of the cosmic censorship conjecture in this specific setting.

The breakthrough of Christodoulou, [4], is concerned with *non-spherical* collapse and black-hole formation for a ``pure" gravitational field, resulting from the focussing of gravitational waves. In particular, it does not require that any field be coupled to the gravitational field through the right-hand-side of (1). Stated informally, the following result says that sufficiently focused ``short" pulse initial data on a null cone will lead to a trapped surface.

Theorem 3 *Let* k *and* l *be positive constants with* $k > 1$ *and* $l < 1$. *Suppose we are given smooth asymptotic initial data at past null infinity that is trivial for advanced time* $u \leq 0$ *and that the incoming energy per unit solid angle in each direction in the advanced time interval* $[0,\delta]$ *is not less than* $k\frac{8}{\pi}$. *Then, if* δ *is suitably small, the maximal development of the data contains a closed trapped surface, of area larger than or equal to* $4\pi l^2$, *diffeomorphic to the unit* 2*-sphere.*

The proof of this result is a monumental tour-de-force in the mathematical analysis of Einstein's equations and a work of tremendous insight that occupies 595 pages of text. One is reminded of the words of Horace, *exegi monumentum aere perennius*.

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Hermitian Symmetry in Several Complex Variables

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The complex numbers do not form an ordered field, and hence the inequalities used in complex analysis (in one or several variables) must involve real numbers. In particular, a complex analytic function defined on a connected open set cannot satisfy an inequality without reducing to a constant. Involving Hermitian symmetry allows us to make sense of inequalities in this context.

We begin with a related mysterious issue that arises in a first course in complex variables. Are z and its complex conjugate $\bar{z}$ independent variables? Of course if one knows z, then one knows $\bar{z}$; thus they *are not* independent. As functions, z and $\bar{z}$ have linearly independent differentials, and hence z and $\bar{z}$ *are* independent in that sense. Consider also the Cauchy-Riemann equation

$$0 = \bar{\partial} h = \sum_{j=1}^{n} \frac{\partial h}{\partial \bar{z}_j} d\bar{z}_j \qquad (1)$$

defining a holomorphic (complex analytic) function h on a domain in complex Euclidean space $\mathbb{C}^n$. Formula (1) states informally that a smooth function is holomorphic if and only if it is independent of each $\bar{z}_j$. This perspective has had a major impact on complex analysis. How can we make sense out of this conundrum?

A polynomial or convergent power series $f(z,\bar{w})$ in $2n$ complex variables is called *Hermitian symmetric* if $f(z,\bar{w}) = \overline{f(w,\bar{z})}$ for all z,w. Using multi-index notation we write $f(z,\bar{w}) = \sum c_{\alpha,\beta} z^{\alpha} \bar{w}^{\beta}$. Let $\mathrm{rk}(f)$ denote the (possibly infinite) rank of the matrix $C = (c_{\alpha,\beta})$. The function f is Hermitian symmetric if and only if the function $z \mapsto f(z,\bar{z})$ is real-valued and if and only if C is self-adjoint. We identify the set of Hermitian polynomials with the ring R of real polynomials in $2n$ variables.

Given $f(z,\bar{z})$, the function $f(z,\bar{w})$ is uniquely determined. The passage from $f(z,\bar{z})$ to $f(z,\bar{w})$ is called *polarization*. This concept illuminates our mystery.

Example 1. A linear transformation on $\mathbb{C}^n$ is *unitary* if it preserves inner products. By polarization, it is unitary if and only if it preserves squared lengths.

Examples of Hermitian symmetric functions arise as follows. Let $\mathcal{H}$ be a Hilbert space with inner product (u,v) and squared norm $||v||^2$. Consider a holomorphic map A with values in $\mathcal{H}$ and define f by $f(z,\bar{w}) = \langle A(z), A(w) \rangle$. We call f a Hermitian squared norm, because $f(z,\bar{z}) = ||A(z)||^2$. More generally, for $\mathcal{H}$-valued holomorphic maps A and B we put $f(z,\bar{w}) = \langle A(z), A(w)\rangle - \langle B(z), B(w)\rangle$ and express $f(z,\bar{z})$ as a difference of Hermitian squared norms.

If Ω is a domain in $\mathbb{C}^n$, h is holomorphic on Ω, and $h(z) \geq 0$ for all z, then h is a constant. For a non-constant Hermitian symmetric function f, however, $f(z,\bar{z}) \geq 0$ for all z is possible. This condition differs from being a squared norm. Comparing these two notions of positivity leads to Hermitian analogues of Hilbert's 17th problem, isometric embedding of holomorphic line bundles, information about proper holomorphic mappings, and insight into the mysterious matter concerning the independence of z and $\bar{z}$. See [D].

We further state a version of Artin's solution to Hilbert's problem and an extension due to Pfister. The number 2^k in Pfister's theorem is independent of the degree of p.

Theorem 1 (Artin) *All values of a polynomial p in k real variables are non-negative if and only if there is a polynomial q, not identically zero, such that $q^2 p$ is a sum of squares of polynomials.*

Theorem 2 (Pfister) *In Artin's theorem, it is always possible to choose q such that $q^2 p$ is a sum of at most 2^k squares.*

Question 1. Let f be a polynomial with $f(z,\bar{z}) \geq 0$. Does the ideal (f) contain a non-trivial Hermitian squared norm? If so, what is the minimum rank?

We seek a (not identically zero) polynomial q, an integer N, and a polynomial map $h : \mathbb{C}^n \to \mathbb{C}^N$ for which $q(z,\bar{z}) f(z,\bar{z}) = ||h(z)||^2 = \sum_{j=1}^{N} |h_j(z)|^2$. By [D] doing so is possible if and only if f is a *quotient of squared norms*. Varolin [V] characterized quotients of squared norms and the relationship with metrics on Hermitian line bundles. By [DL] the Hermitian analogue of Pfister's theorem fails. There is no bound on the minimal N depending only on the dimension.

Lemma 1 *For $n = 1$ set $f(z,\bar{z}) = (1 + |z|^2)^d$. Assume that $||h||^2$ is a multiple of f and that h is not identically 0. Then $\mathrm{rk}(||h||^2) \geq d + 1$.*

Proof. The key step (and all we present here) is the following elementary statement. Let r be a polynomial in one real variable which is divisible by $(1 + x)^d$. Then either r is identically zero or r has at least $d + 1$ terms. The proof uses the method of descent. Find the **smallest** d for which there are polynomials q and r such that $r(x) = (1 + x)^d q(x)$ and r has at most d terms. Either both q are r are divisible by x or neither is divisible by x. We may divide out all factors of x and therefore assume that either r is identically zero or r has a nonzero constant term. In the second situation, differentiate both sides to obtain

$$r'(x) = (1 + x)^{d-1}(dq(x) + q'(x)(1 + x)).$$

Now r' has at most $d - 1$ terms, and it is divisible by $(1 + x)^{d-1}$. We have replaced d with $d - 1$. The conclusion follows by the method of descent.

Question 2. Let x be an algebraic subset of $\mathbb{C}^n$. Let f be a Hermitian polynomial with $f(z,\overline{z}) \geq 0$ on X. Are there an integer N and a polynomial mapping $h : \mathbb{C}^n \to \mathbb{C}^N$ for which $f(z,\overline{z}) = ||h(z)||^2 = \sum_{j=1}^{N} |h_j(z)|^2$ on X? Here are some specific classical answers.

Theorem 3 (Riesz-Fejer) *Let f be a non-negative trig polynomial. Then there is a single polynomial $h(z)$ such that $|h(z)|^2 = f(z)$ on the unit circle.*

Here X is the unit circle, $N = 1$, and h has the same degree as f. Things are more subtle in several variables, where Theorem 4 holds. There are no possible bounds on N or the degree of h depending only on the degree of f and the dimension n.

Theorem 4 (Catlin-D'Angelo) *Let $f(z,\overline{z})$ be a polynomial that is strictly positive on the unit sphere in $\mathbb{C}^n$. Then there are an integer N and a holomorphic polynomial $h : \mathbb{C}^n \to \mathbb{C}^N$ such that $||h(z)||^2 = f(z)$ on the unit sphere.*

The conclusion in Theorem 3 holds for any non-negative f. The conclusion in Theorem 4 may fail if f has zeros. Put $f_a(z,\overline{z}) = |z_1|^4 - a|z_1|^2|z_2|^2 + |z_2|^4$, where $a \leq 2$. When $a < 2$, Theorem 4 applies. The minimal integer N_a and the minimal degree of the mapping h_a tend to infinity as a approaches 2. When $a = 2$, $f_a(z,\overline{z}) \geq 0$ for all z, but f agrees with no Hermitian squared norm on the sphere.

Theorem 5 (Aronszajn) *Let $z \to f(z,\overline{z})$ be real-analytic on an open connected subset Ω of $\mathbb{C}^n$. Then there are a Hilbert space $\mathcal{H}$ and a holomorphic mapping $h : \Omega \to \mathcal{H}$ such that $f(z,\overline{z}) = ||h(z)||^2$ if and only if the following condition holds:*

For every positive integer N, and for every choice of N points $w_1,\ldots,w_N \in \Omega$ the matrix $f(w_j,\overline{w}_k)$ for $1 \leq j,k \leq N$ *is non-negative definite.*

The same idea applies as follows [DP] for real algebraic subsets X. Let I be an ideal in R and suppose $S \subset \mathbb{C}^n$. We say that S is *Hermitian null* with respect to I if, whenever R is a Hermitian polynomial in I, and $p, q, \in S$, then $R(p,\overline{q}) = 0$. Here is a simple example. Let I be the ideal generated by $||z||^{2m} - 1$. Fix p on the unit sphere; let $S = \{\eta^j p\}$ for $0 \leq j < m$, where η is a primitive m-th root of unity. Then S is Hermitian null with respect to I, because $\langle \eta^j p, \eta^k p\rangle^m - 1 = \langle p,p\rangle^m - 1 = 0$.

Theorem 6 *Let r and f be Hermitian polynomials. Assume S is Hermitian null with respect to (r), and $\{z_l\} \subset S$. If f is congruent to a Hermitian squared norm modulo the ideal (r), then the matrix $f(z_j,\overline{z}_k)$ is non-negative definite.*

Example 2. Put $r(z,\overline{z}) = |z_1|^4 + |z_2|^2 - 1$ and $f(z,\overline{z}) = 2 - |z_1|^2$. Then f is positive on the zero-set of r, but there is no polynomial mapping h with $f(z,\overline{z}) = ||h(z)||^2$ on the zero-set of r. Consider the points $p = (1,0)$ and $q = (-1,0)$. Simple calculation shows that (p,q) is Hermitian null, but the two-by-two matrix resulting from f has a negative eigenvalue:

$$\begin{pmatrix} f(p,\overline{p}) & f(p,\overline{q}) \\ f(q,\overline{p}) & f(q,\overline{q}) \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$$

Hermitian symmetry has additional applications to mapping theorems in several complex variables [LP], representation theory, and algebraic combinatorics.

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	8:30 – 9:20 Marius Junge Plenary Lecture | Conférence plenaire		
	9:30 – 10:00 Break | Pause	10:00 – 10:30 Break | Pause	10:00 – 10:30 Break | Pause
	10:00 – 11:30 Scientific Sessions Session scientifiques	10:30 – 11:20 Ulrike Tillmann Plenary Lecture | Conférence plenaire	10:30 – 11:20 Greg Smith Coxeter-James Prize Lecture Conférence Prix Coxeter-James
11:00 AM – 13:00 CMS Development Group Meeting (Regina Inn, Wascana B)	11:30 – 12:20 Veselin Jungic Excellence in Teaching A. Lecture Prix d'excellence en enseignement	11:30 – 12:20 Ailana Fraser Krieger-Nelson Prize Lecture Conférence Prix Krieger-Nelson	11:30 – 12:20 Lisa Jeffrey Plenary Lecture | Conférence plenaire
13:00 – 17:00 CMS Board of Directors Meeting (Regina Inn, Wascana B)	12:30 – 14:00 – Break | Pause NSERC LRP (Nancy Reid)	12:30 – 14:00 – Break | Pause CMS Annual General Meeting	12:30 – 14:00 Break | Pause
	14:00 – 15:00 Scientific Sessions Session scientifiques	14:00 – 15:00 Scientific Sessions Session scientifiques	14:00 – 16:00 Scientific Sessions Session scientifiques
	15:00 – 15:50 Margaret Walshaw Plenary Lecture | Conférence plenaire	15:00 – 15:50 Dror Bar-Natan Plenary Lecture | Conférence plenaire	
	16:00 – 16:15 Break | Pause	16:00 – 16:15 Break | Pause	
	16:15 – 17:45 Scientific Sessions Session scientifiques	16:15 – 17:45 Scientific Sessions Session scientifiques	
18:00-19:00 John Fyfe Public Lecture | Conférence publique (Regina Inn, Kenosee)	18:00-19:30 Malgorzata Dubiel 2011 Prix Adrien Pouliot Award Public Lecture | Conférence publique	18:30 - 19:15 Reception | Réception 19:15 – 22:00 Banquet (Wascana A)	
19:00-20:30 Welcome Reception Réception de bienvenue	20:00 – 22:00 Student Social		updated | mise à jour February 17 fevrier

June 2-4, 2012, Regina Inn and Ramada Hotel (Regina, SK)
Host: University of Regina

2-4 juin, hotels Regina Inn et Ramada (Regina, SK)
Hôte : Université de Regina

Sessions

Cluster Algebras and Related Topics
Algèbres amassées et sujets reliés
Org: Ralf Schiffler (Connecticut), Hugh Thomas (UNB)

Combinatorics | *Combinatoire*
Org: Karen Meagher (Regina), Marni Mishna (SFU)

Complex Geometry and Related Fields
Géométrie complexe et domaines reliés
Org: Tatyana Barron (Western), Eric Schippers (Manitoba)

Computation of Analytical Operators in Applied and Industrial Mathematics | *Calcul des opérateurs analytiques en mathématiques appliquées et industrielles*
Org: Peter Gibson (York), Michael Lamoureux (Calgary)

Connections in Mathematics Education
Connexions dans l'enseignement des mathématiques
Org: Roberta La Haye (Mount Royal), Patrick Maidorn (Regina), Kathy Nolan (Regina)

Free Probability Theory: New Developments and Applications | *Théorie des probabilités libres: applications et développements récents*
Org: Serban Belinschi (Saskatchewan), Benoît Collins (Ottawa)

Geometric Topology | *Topologie géometrique*
Org: Steve Boyer (UQAM), Ryan Budney (Victoria), Dale Rolfsen (UBC)

Geometry and Topology of Lie Transformation Groups | *Géométrie et topologie des groupes de transformation de Lie*
Org: Lisa Jeffrey (Toronto), Liviu Mare (Regina)

Harmonic Analysis and Operator Spaces
Analyse harmonique et espaces d'opérateurs
Org: Yemon Choi (Saskatchewan), Ebrahim Samei (Saskatchewan)

Homotopy Theory | *Théorie de l'homotopie*
Org: Kristine Bauer (Calgary), Marcy Robertson (Western)

Interactions Between Algebraic Geometry and Commutative Algebra | *Intéractions entre la géométrie algébrique et l'algèbre commutative*
Org: Susan Cooper (Central Michigan), Sean Sather-Wagstaff (North Dakota State)

Number Theory | *Théorie des nombres*
Org: Mark Bauer (Calgary), Richard McIntosh (Regina), Eric Roettger (Mount Royal)

Operator Algebras | *Algèbres des opérateurs*
Org: Martin Argerami (Regina), Juliana Erlijman (Regina), Remus Floricel (Regina)

Perspectives in Mathematical Physics
Perspectives en physique mathématique
Org: Yvan Saint-Aubin (Montréal), Luc Vinet (Montréal)

Representation Theory of Groups, Lie Algebras, and Hopf Algebras | *Théorie de représentation des groupes, des algèbres de Lie et de Hopf*
Org: Allen Herman (Regina), Fernando Szechtman (Regina)

Total Positivity | *Positivité totale*
Org: Shaun Fallat (Regina), Michael Gekhtman (Notre Dame)

Contributed Papers | *Communications libres*
Org: Edward Doolittle (First Nations University), Fotini Labropulu (Regina)

AARMS-CMS Student Poster Session
Présentations par affiches pour étudiants
Org: Bahman Ahmadi (Regina), Ruhi Ahmadi (Regina), Yong Liu (Regina)

Workshops | *Ateliers*
May 29 - June 1, University of Regina
29 mai - 1 juin, Université de Regina (en anglais)

Connections Between Algebra and Geometry
Groups, Lie Algebras, and Hopf Algebras

Call for Nominations

The CMS invites nominations for the 2012 **Adrien Pouliot Award**. The award recognizes individuals or teams of individuals who have made significant and sustained contributions to mathematics education in Canada. Such contributions are to be interpreted in the broadest possible sense and might include: community outreach programs, the development of a new program in either an academic or industrial setting, publicizing mathematics so as to make mathematics accessible to the general public, developing mathematics displays, establishing and supporting mathematics conferences and competitions for students, etc.

The deadline for nominations is April 30, 2012. Please submit your nomination electronically, preferably in PDF format, to **apaward@cms.math.ca.**

Nomination requirements:

- Include contact information for both nominee and nominator.
- Describe the nominated individual's or team's sustained contributions to mathematics education. This description should provide some indication of the time period over which these activities have been undertaken and some evidence of the success of these contributions. This information must not exceed four pages.
- Two letters of support from individuals other than the nominator should be included with the nomination.
- Curricula vitae should not be submitted since the information from them relevant to contributions to mathematics education should be included in the nomination form and the other documents mentioned above.
- If a nomination was made in the previous year, please indicate this.
- Members of the CMS Education Committee will not be considered for the award during their tenure on the committee.

Renewals:

Individuals who made a nomination last year can renew this nomination by simply indicating their wish to do so by the deadline date. In this case, only updating materials need be provided as the original has been retained.

Appel des mises en candidature

La SMC sollicite des mises en candidature pour le **Prix Adrien Pouliot** 2012. Le prix récompenser aux personnes ou aux groupes qui ont fait une contribution importante et soutenue à l'enseignement des mathématiques au Canada. Le terme « contribution » s'emploie ici au sens large; les candidats pourront être associés à une activitée de sensibilisation, un nouveau programme adapté au milieu scolaire ou à l'industrie, des activités promotionnelles de vulgarisation des mathématiques, des initiatives, spéciales, des conférences ou des concours à l'intention des étudiants, etc.

La date limite pour des mises en candidature est le 30 avril 2012. Veuillez faire parvenir votre mise en candidature par voie électronique, de préférence en format PDF, à **prixap@smc.math.ca.**

Conditions de candidature :

- Inclure les coordonnées du/des candidats ainsi que le(s) présentateur(s).
- Décrire en quoi la personne ou le groupe mise en candidature a contribué de façon soutenue à des activités mathématiques. Donner un aperçu de la période couverte par les activités visées et du succès obtenu. La description ne doit pas être supérieur à quatre pages.
- Le dossier de candidature comportera deux letters d'appui signées par des personnes autres que le présentateur.
- Il est inutile d'inclure des curriculums vitae, car les renseignements qui s'y trouvent et qui se rapportent aux activités éducatives visées devraient figurer sur le formulaire de mise en candidature et dans les autres documents énumérés ci- dessus.
- Si la mise en candidature a été soumise en l'année précédente, s'il vous plaît indiquez-le.
- Les membres du Comité d'éducation de la SMC ne pourront être mise en candidature pour l'obtention d'un prix pendant la durée de leur mandat au Comité.

Renouveler une mise en candidature :

Il est possible de renouveler une mise en candidature présentée l'an dernier, pourvu que l'on en manifeste le désir avant la date limite. Dans ce cas, le présentateur n'a qu'à soumettre des documents de mise à jour puisque le dossier original a été conservé.

Call for Nominations

The CMS Research Committee is inviting nominations for three prize lectureships. These prize lectureships are intended to recognize members of the Canadian mathematical community.

The Coxeter-James Prize Lectureship recognizes young mathematicians who have made outstanding contributions to mathematical research. The recipient shall be a member of the Canadian mathematical community. Nominations may be made up to ten years from the candidate's Ph.D: researchers having their PhD degrees conferred in 2002 or later will be eligible for nomination in 2012 for the 2013 prize. A nomination can be updated and will remain active for a second year unless the original nomination is made in the tenth year from the candidate's Ph.D. The prize lecture will be given at the 2013 CMS Winter Meeting.

The Jeffery-Williams Prize Lectureship recognizes mathematicians who have made outstanding contributions to mathematical research. The recipient shall be a member of the Canadian mathematical community. A nomination can be updated and will remain active for three years. The prize lecture will be given at the 2013 CMS Summer Meeting.

The Krieger-Nelson Prize Lectureship recognizes outstanding research by a female mathematician. The recipient shall be a member of the Canadian mathematical community. A nomination can be updated and will remain active for two years. The prize lecture will be given at the 2013 CMS Summer Meeting.

The deadline for nominations is June 30, 2012.

Nominators should ask at least three referees to submit letters directly to the CMS by September 30, 2012. Some arms-length referees are strongly encouraged. Nomination letters should list the chosen referees, and should include a recent curriculum vitae for the nominee, if available. Nominations and reference letters should be submitted electronically, preferably in PDF format, by the appropriate deadline to the corresponding email address:

Coxeter-James: **cjprize@cms.math.ca**
Jeffery-Williams: **jwprize@cms.math.ca**
Krieger-Nelson: **knprize@cms.math.ca**

Appel de mises en candidature

Le Comité de recherche de la SMC lance un appel de mises en candidatures pour trois de ses prix de conférence. Ces prix ont tous pour objectif de souligner l'excellence de membres de la communauté mathématique canadienne.

Le prix Coxeter-James rend hommage aux jeunes mathématiciens qui se sont distingués par l'excellence de leur contribution à la recherche mathématique. Cette personne doit être membre de la communauté mathématique canadienne. Les candidats sont admissibles jusqu'à dix ans après l'obtention de leur doctorat : ceux qui ont obtenu leur doctorat en 2002 ou après seront admissibles en 2012 pour le prix 2013. Toute mise en candidature est modifiable et demeurera active l'année suivante, à moins que la mise en candidature originale ait été faite la 10e année suivant l'obtention du doctorat. La personne choisie prononcera sa conférence à la Réunion d'hiver SMC 2013.

Le prix Jeffery-Williams rend hommage aux mathématiciens ayant fait une contribution exceptionnelle à la recherche mathématique. Cette personne doit être membre de la communauté mathématique canadienne. Toute mise en candidature est modifiable et demeurera active pendant trois ans. La personne choisie prononcera sa conférence à la Réunion d'été SMC 2013.

Le prix Krieger-Nelson rend hommage aux mathématiciennes qui se sont distinguées par l'excellence de leur contribution à la recherche mathématique. La lauréate doit être membre de la communauté mathématique canadienne. Toute mise en candidature est modifiable et demeurera active pendant deux ans. La lauréate prononcera sa conférence à la Réunion d'été SMC 2013.

La date limite de mises en candidature est le 30 juin 2012.

Les proposants doivent faire parvenir trois lettres de référence à la SMC au plus tard le 30 septembre 2012. Nous vous incitons fortement à fournir des références indépendantes. Le dossier de candidature doit comprendre le nom des personnes données à titre de référence ainsi qu'un curriculum vitae récent du candidat ou de la candidate, dans la mesure du possible. Veuillez faire parvenir les mises en candidature et lettres de référence par voie électronique, de préférence en format PDF, avant la date limite, à l'adresse électronique correspondante:

Coxeter-James: **prixcj@smc.math.ca**
Jeffery-Williams: **prixjw@smc.math.ca**
Krieger-Nelson: **prixkn@smc.math.ca**