CMS NOTES

The Two Cultures of Mathematics

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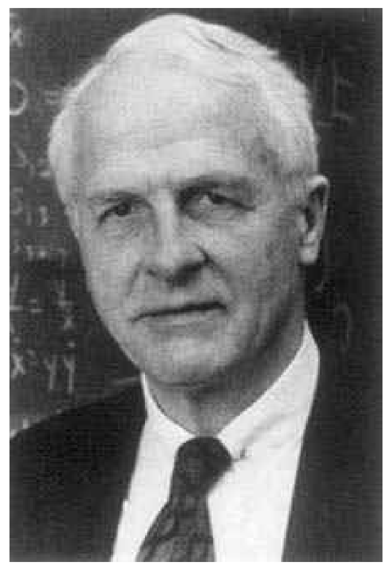
Like the two cultures of the sciences and the humanities, as lamented by C. P. Snow in his influential and controversial Rede Lecture [4], there are two cultures within the mathematics community itself. Snow used specific examples of cultural clashes to illuminate his argument, and similarly here I will attempt to shed some light on the different cultures of pure and applied mathematics by recounting a 1950–51 conflict involving Harvard mathematician Garrett Birkhoff and J. J. Stoker, one of the founders of the Courant Institute of Mathematical Sciences.

Garrett Birkhoff, as most readers will know, co-authored (with Saunders Mac Lane) the classic text, *A Survey of Modern Algebra*. However, during and after the Second World War, he began working in more applied areas. I first encountered his monograph on fluid dynamics [1] when I was exploring Euler's role in the history of d'Alembert's paradox [2]. As an applied mathematician who had done research in fluid flow (and later developed an interest in the history of mechanics), I was struck by the fresh approach and clarity of Birkhoff's writing on the subject. The entire first chapter [1, pp. 3–39] of his monograph is devoted to paradoxes of fluid flow; d'Alembert's paradox is the first discussed [1, pp. 10–13].


Garrett Birkhoff, *MacTutor*.

As Birkhoff explained, this paradox involves the steady, uniform flow of a non-viscous, incompressible fluid (often called an ideal fluid) past a smooth, finite body (such as a sphere). Flows of this kind can be described by the gradient of a potential function, which implies the flow exerts no drag on the body, a result contradicted by the physical fact of substantial drag exerted by actual flows. For Birkhoff, d'Alembert's paradox and others are "in part at least, paradoxes of topological oversimplification and symmetry paradoxes" [1, p. 22]. Euler's resolution of d'Alembert's paradox challenged the assumption of an incompressible fluid (especially for a ball shot through the air) [2]; two other plausible resolutions are considered below.

One topological oversimplification associated with "the hypothesis of an 'ideal fluid'" is "that a locally single-valued velocity potential *U* is single-valued in the large" for two-dimensional flow past a body. To avoid symmetry paradoxes, Birkhoff advised the reader to "admit the possibility that *a symmetrically stated problem may not have any stable symmetric solution*"; in the case of the uniform flow of an ideal fluid past a sphere, for example, a steady, axially-symmetric solution exists mathematically, but "there is no reason to suppose that any steady flow is *stable*." An instability makes a steady, mathematical flow physically unrealizable, and "irregular, turbulent 'eddies' … in the 'wake' of an obstacle" might therefore occur in actual flows [1, pp. 20–21; italics are Birkhoff's]. An instability of this kind might thus resolve d'Alembert's paradox.

Birkhoff opined that theories of fluid dynamics can be learned "more effectively … by studying the paradoxes" he described. He criticized textbooks that attributed the gap between theory and experiment to the difference between real fluids with "small but finite viscosity" and ideal fluids of "zero viscosity," and he thought "that to attribute them all [the paradoxes he describes] to the neglect of viscosity is an unwarranted oversimplification"—the "root lies deeper, in lack of precisely that deductive rigor whose importance is so commonly minimized by physicists and engineers" [1, pp. 3–4]. The paradoxes warn against "the impression … that mathematical deduction should be supplanted by 'physical' reasoning," which can lead to flawed approximations and oversimplifications, though Birkhoff admitted the usefulness of "oversimplifications based on the 'right' approximations." He continued, "mathematicians can perform a useful service if they will analyze critically these oversimplifications, by the deductive method, and so establish their limitations more clearly" [1, p. 37]. Birkhoff offered his paradoxes to a subject that is primarily the domain of engineers and applied mathematicians (like me, in my past life as a fluid dynamics specialist). His criticisms were severe and, in fact, he named J. J. Stoker as one who "effectively exploited" an "analogy" between two kinds of waves, even though another paradox (not d'Alembert's) made one kind "*mathematically impossible*" [1, pp. 22–24; italics are Birkhoff's].

As life and luck would have it, J. J. Stoker wrote a review [5] of Birkhoff's 1950 monograph. Stoker's assessment of Birkhoff's Chapters 2 through 5 was balanced, even complimentary in the cases of Chapter 2 (on problems with free boundaries) and Chapter 3 (on modelling and dimensional analysis). Stoker's review of Chapter 1, however, was withering. He found "it difficult to understand for what class of readers the first chapter was written"; indicated

J. J. Stoker, ca 1960, Courant Institute

that "the majority of cases cited as paradoxes" were either "mistakes long since rectified" or "discrepancies between theory and experiment the reasons for which are also well understood"; and worried that "the uninitiated would be very likely to get wrong ideas about some of the important and useful achievements in hydrodynamics from reading this chapter." Referring to "some general observations regarding the philosophy and correct attitude toward applied mathematics" made by Birkhoff, Stoker allowed that most "workers in the field would agree quite well with the author's observations," but he thought that "they are perhaps better informed in some cases than the author would seem to imply" [5, pp. 497–498].

To illustrate this last point, Stoker offered salient mathematical reasoning underlying the generally accepted resolution of d'Alembert's paradox: "the small coefficients involving viscosity occur in terms containing derivatives of the highest order in the system of differential equations, and thus developments in the neighborhood of zero viscosity involve boundary layer effects because of the loss of order of the differential equations in the limit." Stoker was referring to the fact that the Navier-Stokes equations, which describe viscous, incompressible flow past a body, are second-order partial differential equations that permit the so-called no-slip condition (that the fluid's velocity vanish on the body's boundary) to be satisfied; in the case of zero viscosity, the Navier-Stokes equations become Euler's equations, which are first-order and allow the flow to be described by a potential function but permit the vanishing of only the flow velocity normal to the body's boundary; the transition from viscous flow (with the no-slip condition on the boundary) to non-viscous flow (with the corresponding loss of second-order terms) farther from the boundary occurs in what is known as a boundary layer, in which the fluid's velocity is approximated mathematically using matched asymptotic expansions (or computationally using a very fine mesh).

Birkhoff had anticipated this persuasive argument from applied mathematics "in support of the view that the paradoxes of fluid mechanics are due to an unjustified neglect of viscosity." He conceded that the argument had "some merit" but thought that it was "inconclusive." For him, "the real question is, why does separation of the boundary layer occur?" This question alluded to observations that the boundary layer adjacent to a body immersed in a flow often separates from that body (downstream from where it begins) to become the border of a turbulent wake behind the body. Birkhoff believed this question "concerns the *stability* of nearly non-viscous flows" [1, p. 27; italics are Birkhoff's]. It seems that Stoker's argument was inconclusive for Birkhoff because it failed to rule out, deductively, the stability question raised by d'Alembert's paradox and others.

Birkhoff, as one steeped in the culture of pure mathematics, saw his paradoxes as guidelines to sharpen the deductive skills of fluid dynamics researchers. Stoker rejected the applicability of those guidelines. In the second (1960) edition of Birkhoff's monograph, the first chapter grew to two chapters; these chapters doubled down on paradoxes (one covered those of non-viscous flow, the other viscous flow), but Birkhoff's earlier criticisms of physicists, engineers, their lack of deductive rigor, and J. J. Stoker were removed. Further, Birkhoff made no suggestion that an instability in the flow of an ideal fluid might resolve d'Alembert's paradox. I prefer to believe that these omissions are evidence of Birkhoff's attempt to reconcile with those working within the culture of applied mathematics. Ironically, however, recent (not yet mainstream) research [3] indicates that Birkhoff's suggested resolution to d'Alembert's paradox might be the best one.

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