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## Bourbaki, Structuralism, and Categories

**Jean-Pierre Marquis** (Université de Montréal)

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**Amy Ackerberg-Hastings**, *chercheuse indépendante (aackerbe@verizon.net)*
**Hardy Grant,** *York University [retraité] (hardygrant@yahoo.com)*

> Since Hilbert and Dedekind, we have known very well that large parts of mathematics can be developed logically and fruitfully from a small number of well-chosen axioms. That is to say, given the bases of a theory in an axiomatic form, we can develop the whole theory in a more comprehensible way than we could otherwise. This is what gave the general idea of the notion of mathematical structure. Let us say immediately that this notion has since been superseded by that of category and functor, which includes it under a more general and convenient form. It is certain that it will be the duty of Bourbaki … to incorporate the valid ideas of this theory in his works [1].

Jean Dieudonné was one of the most eminent French mathematicians of the 20th century. The foregoing quote was written in 1970, when Dieudonné and his colleagues were reflecting on Bourbaki's legacy. The claims made by Dieudonné are puzzling: on the one hand, mathematicians had by then a general idea of the notion of mathematical structure; on the other hand, category theory superseded that notion and Bourbaki, whoever that is, must incorporate the valid ideas of this theory in his work since they apparently give a better account of the notion of mathematical structure.

In the 1970s, every mathematician knew who Bourbaki was and what he had done. There were fans—big fans—and there were critics—virulent critics. Nowadays, not so much. Very few, I suspect, could say what the general idea of the notion of mathematical structure is precisely, or explain that Bourbaki defended a structuralist view of pure mathematics. Meanwhile, nowadays every mathematician, young and old, has at least heard of categories and functors. Was Dieudonné right? Did Bourbaki ever integrate categories and functors in his work? The short answer is 'no'. It is true that Bourbaki had a general idea of the notion of mathematical structure. But Bourbaki, although he knew about categories and functors from very early on, never found a way to integrate them in his work. With hindsight, this failure should not be surprising, for the issue is delicate and subtle. It brings us directly to the center of some of the most sensitive issues in the foundations of mathematics.

So, who is Bourbaki? What was his idea of the notion of mathematical structure? And what do categories and functors have to do with it? Let us take a quick look.

### Bourbaki: a nano-historical sketch

Our story begins in Paris, December 1934, Boulevard Saint-Michel, in a small café [2]. A group of young university professors, mathematicians and almost all former students of the École Normale Supérieure, an elite school in France, sit around a table, discussing the appalling state of introductory textbooks in analysis available in French at the time. They decide that the only way to remedy the situation is to write a new textbook, a *modern* textbook, in the spirit of Van der Waerden's *Moderne Algebra* [3], a work they admire. So, they sketch the plan of the book and decide that they first need to write an introduction to the new "abstract" mathematics—that is, set theory, algebra, and topology—before moving on to analysis proper. They decide to join forces and meet in the following months to write the work. Thus, Bourbaki was born; although still nameless and yet to be fully organized, this improbable collaborative project was about to change the face of contemporary mathematics to this day.

**Figure 1.** Cartan, de Possel, Dieudonné, Weil (standing), Mirlès, Chevallez, and Mandelbrojt (seated) at first official meeting of the Bourbaki group in 1935. MacTutor

Around the table were: Jean Dieudonné, André Weil, Henri Cartan, Claude Chevalley, Jean Delsarte, Jean Coulomb, René de Possel, Charles Ehresmann, and Szolem Mandelbrojt, nowadays called "the founding fathers" of Bourbaki [4]. They all had international careers as creative mathematicians. The first official meeting of the group happened in the summer of 1935. This is where they decided to adopt the name « N. Bourbaki » (later to become « Nicolas Bourbaki »), an absurd joke, typical of the humor practiced by the students of the École Normale Supérieure at the time.

The project, limited at first to an introductory volume on analysis, morphed into a vast and ambitious enterprise: the idea was to start from scratch, to present and develop the abstract axiomatic theories needed to get to classical analysis. Nothing is taken for granted, nothing is presupposed except for a certain "mathematical maturity", whatever that is. The mathematics should all be developed by purely logical means. It is, in a sense, an exercise in formalization, abstraction, and organization, although it is certainly not seen that way. Thus, the first book starts with an exposition of first order logic, then an axiomatic presentation of set theory and the general idea of structure. Subsequent volumes move on to the axioms of topology, the axioms for algebraic structures (e.g., monoids, groups, rings, fields). The next step consists in combining these structures (e.g., topological groups, topological vector spaces), and all this eventually brings us to analysis at last.

The written style is terse: definitions, theorems, proofs. Nothing else. No discussion, no image, no context. There are exercises and problems—a lot of them—and they are seen as an integral part of the presentation. The proofs are not written in the formal system—which the members see as a long and tedious exercise—but it is explicitly assumed that any competent mathematician should be able to translate the given proofs into the formal jargon. To this day—for Bourbaki still exists and still holds a seminar in Paris—more than 40 volumes have been published [5]. An introductory textbook? Absolutely not. An encyclopedia? The enterprise was never conceived as one and it certainly is not such a work. It is not quite clear what he ended up with. It is a singular object. But it influenced at least three generations of mathematicians worldwide. Emil Artin, Philip Hall, Samuel Eilenberg, and Saunders Mac Lane all praised the first volumes published in the 1940s. Michael Atiyah later said that his generation were all "bourbakistes". Be that as it may, at the core of the whole project is one key idea, one motivating *leitmotiv*: (pure) mathematics is about *abstract structures*.

### Bourbaki's notion of (species of) structure and structuralist mathematics

What is, *in general*, a mathematical structure? Already in 1935, inspired by the recent developments in algebra, topology, and logic, Bourbaki decided that his project had to be based on the idea of mathematical structure. But how does one proceed to define *in general* the notion of a mathematical structure? How does one spell out concretely and rigorously the idea that all of mathematics is based on abstract structures? Bourbaki literally struggled for more than twenty years with these questions before he finally published his answer, even though at the end he knew very well that it did not work, in particular it could not accommodate categories and functors. But since he could not find a better way of doing it and many volumes that were based on his idea of mathematical structure had already been published, he had to let it go.

Nowadays, one thinks of a mathematical structure as a set with relations and/or operations satisfying certain conditions. Bourbaki's approach contains these elements, but his presentation is different. First, he defines what he called an echelon construction. Simplifying greatly, one can say that an echelon construction, denoted by $E$, is a basically a way to construct, from given basic sets, a new set by combining cartesian products and powersets on the given basic sets in a certain order. More precisely, one starts with basic sets $A_1,\ldots, A_n$, and parameter sets $B_1,\ldots, B_m$, and then proceeds by recursion, i.e., if $X$ and $Y$ have been constructed, that is, if they are in the echelon $E$, then the cartesian product $X \times Y$ and the powerset $\wp(X)$ are also in the echelon $E$, then the set obtained at the end of such a sequence of constructions is an echelon construction $E$.

The second step consists in defining a *species of structure* as one would expect, that is, by introducing relations and conditions on elements of a given echelon construction. We seem to get something akin to our original informal characterization, with the only difference that the underlying sets required are constructed appropriately. For instance, a topology on a set $A$, seen as a Bourbakian structure, is a set $O_A$ in the echelon construction $\wp(\wp(A))$ satisfying the usual conditions on open sets. Thus, one defines relations $R_1,\ldots, R_k$ on elements of the echelon construction $E$ to get to a species of structure. But that is not all. There is one missing ingredient that captures the essence of abstract mathematical structuralism: the relations must be *transportable*. This basically means that isomorphisms between two lists of basic sets should yield "the same" species of structures and thus that any theorem proven for one specific structure must hold for any structure isomorphic to the first one. Today, we would say that everything is done up to isomorphism [6].

Our foregoing presentation has been entirely informal. When one looks at Bourbaki's work, one immediately notices that he is firmly in a *formal* framework. More precisely, he is working in a formal language for set theory. He even says that a species of structure is a *text*. He does not give a *mathematical* definition of structure, but a *metamathematical* one. It was only natural for Bourbaki to use formal language for set theory to define species of structure in the 1930s and 1940s. However, by 1950, there was one type of mathematical structure that could not be reconstructed as a species of structure, namely categories.

### Categories and Bourbaki's species of structure

In 1950, Samuel Eilenberg joined Bourbaki while he was collaborating with Henri Cartan on their then-forthcoming book *Homological Algebra* [7]. As is well-known, homological algebra treats certain functors between certain types of categories. There is no need to be more precise here. It was clear to everyone at the time that certain categories—e.g., the category of sets or the category of abelian groups or the category of modules over a ring $R$—cannot be sets without imposing some restrictions. Thus, categories in general cannot be set-based structures, for they do not have underlying sets, and their structure does not lend itself to the usual echelon construction and species of structure. One *could* move to a theory of classes, as Eilenberg and Mac Lane did in their original paper, but that was of no use to Bourbaki. Eilenberg was asked to help. Ralf Krömer found interesting passages in Eilenberg's *Nachlass* that exhibit the latter's struggle [8]:

> The method of functors and categories is in some sort of "competition" with the method of structures as developed at present. Unless this "competition" is resolved only one of these methods should be presented at the early stage. Bourbaki is committed by structures for all the material of part I at least [quoted in 8, p. 142].

It seems that Eilenberg's first idea to resolve the conflict was to subsume the notion of structure under the notions of categories. We read:

> The resolution of this "competition" is only possible through the definition of the notion of "structural homomorphism" which would convert each type of structure into a category [quoted in 8, p. 142].

But putting categories first also required that every type of structure comes with a notion of morphism, and some of the members of the group—especially André Weil—did not think this was justified. In a sense, it was simply too early to see clearly what the real problems were, for some of the basic concepts of category theory had still not seen the light! Indeed, the notions of equivalence of categories, adjoint functors, functor categories, representable functors, etc., were officially introduced between 1957 and 1961, after Bourbaki's volume on sets, structures, and isomorphisms.

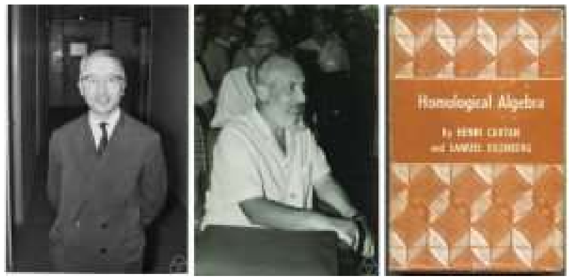


**Figure 2.** Cartan in 1968, Eilenberg in 1970, and *Homological Algebra* in 1956. CC BY-SA 2.0 de

Nonetheless, it was still possible to introduce categories and functors later in the enterprise, e.g., in the volumes on homological algebra, or algebraic topology, or in fact, in any other volume where categories came to play a central role. But Bourbaki did not do it and, in that respect, he failed.

By the late 1950s, Bourbaki was deeply divided, to the point that Alexandre Grothendieck, then a member of Bourbaki, left the group over the issue. Pierre Cartier has reported that Weil and Grothendieck were not even talking to each other [9].

**Sets, categories and structures**

However, to see a competition between Bourbaki's method of structures and categories is to confuse the issue, although unfortunately this is still a prevalent opinion, among certain philosophers of mathematics, for instance [10]. Bourbaki could not have articulated properly how his structuralist standpoint could accommodate category theory, for the latter *extends* his standpoint in a new conceptual direction. It does not *compete* with it, which is what most members seemed to believe even at the end of the last century.

To understand how Bourbaki's view on mathematical structures can accommodate categories, even higher-dimensional categories, is no trivial matter. One must rethink the foundations of mathematics from the ground up, from the formal apparatus to the semantics universe. Although we still do not have a complete picture of this new universe, we have candidates that point in precise directions, namely homotopy type theory and Michael Makkai's FOLDS [11]. One thing is sure: Bourbaki's metamathematical analysis of mathematical structuralism still stands, in the sense that within the forthcoming foundational framework, one will have to build in a requirement that the relations defining the structures have to be transportable with respect to an appropriate notion of isomorphism. This latter constraint being purely *metamathematical*, one does not have to give a general *mathematical* definition of structure; rather, one must specify the formal constraints within which such theories have to be constructed—and those were essentially identified by Bourbaki himself. It might still be possible to be a structuralist with respect to pure mathematics.

*Jean-Pierre Marquis teaches logic, philosophy of science and philosophy of mathematics at the Université de Montréal. He has published a book on category theory and categorical logic as well as numerous articles in logic, philosophy of science, philosophy and the foundations of mathematics.*

## Notes

[1] Dieudonné, J. (1970) The Work of Nicholas Bourbaki. The American Mathematical Monthly 77(2), 134–145, on p. 138.

[2] For more on Bourbaki, see Mashaal, M. (2002) *Bourbaki: une société secrète de mathématiciens*, Les génies de la science no 2. Paris: Pour la Science. English translation by Anna Pierrehumbert (2006) American Mathematical Society.

[3] Waerden, B. L. van der. (1930–1931) *Moderne Algebra*. 2 vol. Berlin: Springer-Verlag.

[4] They might not all have been around the table at this first informal meeting. But they joined the group at one point or another and these people are nowadays considered to be the "founding fathers".

[5] The exact number depends on various parameters, e.g., language, editors, editions, etc.

[6] For the formal details, see Bourbaki, N. (2004) *Theory of Sets*, Springer. The relevant chapter is called "Structures and Isomorphisms"; it was originally published in 1957 and then integrated within a complete book in 1970.

[7] Cartan, Henri, and Samuel Eilenberg. (1956) *Homological Algebra*. Princeton Mathematical Series no 19. Princeton: Princeton University Press. Reprinted in 1999.

[8] For more details, see Krömer, R. (2006) La 'Machine de Grothendieck' se fonde-t-elle seulement sur des vocables métamathématiques? Bourbaki et les catégories au cours des années cinquante, *Revue d'histoire des mathématiques* 12, 119–162.

[9] Cartier, P. (2015) Alexander Grothendieck: A Country Known Only by Name. *Notices of the AMS* 62(4), 373–382, on p. 375.

[10] See Hellman, G. (2003) Does Category Theory Provide a Framework for Mathematical Structuralism? *Philosophia Mathematica*, Series III, 11(2), 129–157.

[11] For homotopy type theory, see Homotopy Type Theory: Univalent Foundations of Mathematics (2013) The Univalent Foundations Program, IAS, Princeton; for Makkai's FOLDS, see Makkai, M. (1998) Towards a Categorical Foundation of Mathematics, in *Logic Colloquium '95 (Haifa)*, edited by J.A. Makowsky & E.V. Ravve, Springer, 153–190.