CMS NOTES

# The Solitude of the Twin Primes

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Over the past twelve or so years, two books have taken the Italian literary scene by storm: "L'amica geniale" ("My brilliant friend") by Elena Ferrante and "La solitudine dei numeri primi" ("The solitude of the prime numbers") by Paolo Giordano.
I read the first book while spending a month in a Tuscan vineyard close to San Gimignano. The protagonists of this novel are two girls, Lila and Len, who live in a neighbourhood of Naples, that had been impoverished long before the war. Lila is a brilliant, excellent student, who dreams of becoming a writer and escaping poverty, while Lenù tries to keep up with her friend, to no avail. The book ultimately became a tetralogy, which was made into an acclaimed television series produced by HBO and RAI. Elena Ferrante's identity is unclear; apparently it is the pseudonym of a writer who wants to safeguard his or her privacy.

On the other hand, "La solitudine dei numeri primi" is the story of two young people, Alice and Mattia, each with serious personal problems, who connect and disconnect romantically over the course of their lives, two similar souls who are nonetheless unable to establish a more permanent relationship. Paolo Giordano wrote the book in 2008 while finishing a Ph.D. thesis in Physics at the University of Turin. The book was a tremendous success, quickly selling more than a million copies, earning the author the prestigious "Strega" Literary prize, and was made into a full-length movie. Actually, the original title of Paolo Giordano's book was "Dentro e fuori l'acqua" (Inside and outside the water); the title "La solitudine dei numeri primi" was suggested by Antonio Franchini (editor of the Mondadori company, publisher of the book) as a sort of publicity stunt. I found the title of Giordano's book quite intriguing: what does it mean to say that "prime numbers are *solitary* "? Yes, Alice and Mattia could come as close as possible, but never get together; to my taste that sounded more like twin primes rather than just primes. While pondering the question I came across a video on YouTube featuring my friend Piergiorgio Odifreddi delivering a talk at the University of Turin, in which he tries to give a mathematical definition of "solitary primes" inspired by the title of Giordano's book.

The Greek mathematician and philosopher Euclid (also known as Euclid of Alexandria) lived in Alexandria around the year 300BC; he wrote a collection of 13 books entitled "Elements", largely a compilation of results by other earlier mathematicians. There we find a proof of the existence of infinitely many prime numbers and a proof of the *Fundamental Theorem of Arithmetic*, namely that any positive integer is a product of powers of prime numbers in a unique way except for ordering. So, we can say that primes are the building blocks of the whole number system.

Now Pythagoras of Samos enters. He was born in 570BC and knew a great deal about numbers and geometry. Pythagoras had several disciples, and together they formed a kind quasi-religious sect, whose members were supposed to guard with their lives the results they obtained. The Pythagoreans discovered (or at least knew) the "Pythagorean formula" for right triangles. The proof, easily illustrated with a picture, was already known to the Babylonians and Indians.

The Pythagoreans noticed that a triangle with sides of length 1 would have a hypotenuse of length $\sqrt{2}$. This was clearly not an integer, but was it a fraction? A member of the group named Hippasus found the proof by contradiction that $\sqrt{2}$ was not a rational number, a proof still taught today. Legend has it that this was not popular with the brethren, and Hippasus was invited to a journey on a boat, thrown overboard, and drowned. The number $\sqrt{2}$ is the first example of an *irrational number*.

Piergiorgio Odifreddi defines a set of numbers to be *solitary* if the series of its reciprocals is convergent. This definition is not empty: the powers of $2$ form a solitary set. Take the sequence $G = \{2^n : n \in \mathbb{N}\}$; the series of its reciprocals $\sum_{n=0}^{\infty} 2^{-n} = 1 + 1/2 + 1/4 + 1/8 + \cdots$ is geometric with ratio $1/2$, hence sums to 2.

The classical example of a nontrivial divergent series is the *harmonic series* $\sum_{n=1}^{\infty} 1/n$ . The name derives from the *harmonics* or *overtones* in music: the wavelengths of the harmonics of a vibrating string are $1, 1/2, 1/3, 1/4, \cdots$ of the string's fundamental wavelength. Pythagoras was active in this field. Although the Greeks certainly knew the harmonic series, it seems that the first convincing proof of its divergence was given by Nicholas Oresme (1325?-1382), Bishop of Lisieux in France. Oresme concentrated on those terms of the harmonic series whose denominator is a power of $2$; he noticed that between $1/2^p$ and $1/2^{p+1}$ there are exactly $2^p$ factors; replacing each of these terms by $1/2^{p+1}$ and adding, we have $1/2$. Since the factors $1/(2^p+1), \cdots 1/(2^{p+1}-1)$ are larger than $1/2^{p+1}$ their sum is larger than $1/2$. Let us give two examples:

$$(1/3 + 1/4) > (1/4 + 1/4) = 1/2;$$

$$(1/5 + 1/6 + 1/7 + 1/8) > (1/8 + 1/8 + 1/8 + 1/8) = 1/2.$$

Proceeding in this way, we conclude that the harmonic series summed from $1$ to $2^n$ is larger than $1/2 + 1/2 + \cdots + 1/2 = n/2$. Since the latter series diverges as $n \to \infty$, so does the former. The manuscript with Oresme's proof was lost, but eventually it was reconstructed in the seventeenth century by Pietro Mengoli. As a result of this result (also still widely taught today!) the set of integers is not solitary according to Odifreddi's definition. In 1881 the mathematician C.W. Merrifield proved that the series of the reciprocals of prime numbers is divergent; his work was published in the Proceedings of the Royal Society of London, vol. 33 (1881-1882). Thus – alas! – neither are the prime numbers solitary.

Odifreddi, however, rather than abandon his definition, shifted his consideration to the *twin* primes: primes that come in pairs $\{p, p+2\}$. It is not known if there are infinitely many such pairs, and the "twin prime conjecture" that there are is a longstanding and stubborn problem in number theory. As a step toward this, the Norwegian mathematician Viggo Brun (1885-1978) proved that the sum of the reciprocals of the twin primes is a constant $B = 1.90216054\cdots$, known as the *Brun Constant*. His proof was published in the paper "La série [lates]\frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \frac{1}{17}[/latex] + où les dénominateurs sont nombres premiers jumeaux est convergente ou finie" , Bull.Sci. Math.,(2) 43, 1919, pages 100-104 and 124-128. The Brun constant $B$ can be an irrational number only if there are infinitely many twin primes. In either case, this shows that twin primes are solitary, according to Odifreddi's definition.

As for the prime numbers themselves, a non-twin prime is called "isolated" (Richard L. Francis, "Isolated Primes", J. Rec. Math., 11 (1978), 17-22.) An easy corollary of Brun's result is that almost all prime numbers have this property. But we must concede that the title "The solitude of the prime numbers" suggested by Giordano's editor is a catchy one. Neither "The solitude of the twin prime numbers" nor "The isolation of most prime numbers" would be as good!