Quaternions at Twilight: Remembering Mary Somerville 150 years after her death

Notes de la SCHPM

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November 29, 2022, marks 150 years since the death of Mary Somerville (1780–1872), one of the most iconic mathematicians of 19th-century Britain. Somerville built her reputation in the drawing rooms of Edinburgh, London, and Paris, in which she demonstrated her expert knowledge of the analytical mathematics recently developed in France. This mathematics was widely believed to be the answer to a long-running decline in British science, owing to the fruitful applications of methods from the differential calculus and the calculus of variations. Somerville's earliest publications were solutions to puzzles printed in *The New Series of the Mathematical Repository*, through which she was one of the earliest adopters of the differential notation in Britain. The work for which she is most famous is undoubtedly her 1831 *Mechanism of the Heavens*, a translation of Laplace's formative *Traité de Mécanique Céleste*, in which he gave an analytical, algebraic treatment of physical astronomy.

Mary Somerville's mathematical works were the focus of my doctoral thesis, which sought to understand what it meant for Somerville, as a woman in 19th-century Europe, to be a mathematician [8]. In early 2022, I was delighted to again have access to the Somerville papers at the Bodleian Library, which I had been unable to consult for the final 18 months of my PhD owing to the Covid-19 pandemic. I especially wanted to resolve the footnotes in my thesis that said, "I know relevant material exists, I just haven't been able to look at it yet!" According to the online catalogue, one folder held a "notebook containing notes and comments in Mary's hand on The Mechanism of the Heavens, n.d. (c.1831)". Much to my surprise, this notebook was accompanied by 70 loose sheets in the shaky handwriting typical of Somerville much later in her life and full of algebraic calculations. One page in particular took my breath away: a draft page from her autobiography, again featuring algebraic formulas down the side, in which she acknowledged that the end of her life was approaching but noted that she was perfectly content in the care and company of her beloved daughters [4]. I was immediately hooked, and could clearly picture Somerville at her writing table, expressing her gratitude that her "intellect [was] still unimpaired", before turning to mathematics, a subject that she had enjoyed for nearly eighty years. I needed to know more about how and why Somerville returned to serious mathematical study when she was nearly ninety.

Figure 1. Self-Portrait of Mary Somerville, undated. Reproduced with the kind permission of the Principal and Fellows of Somerville College, Oxford.

By 1870, when this tale begins, Somerville had been living a peripatetic life on the Italian peninsula for around thirty years. Moving to Italy had been decided upon for health reasons, to seek a better climate than the UK offered, and also in hope of living at a lower cost than in London. During this lengthy sojourn, Somerville published multiple highly successful books—giving surveys of recent scientific developments in the physical sciences and physical geography—yet she still felt on the outside of the scientific community. Individuals such as Giovanni Plana, Professor of Astronomy at the University in Turin, offered her access to their personal libraries, but this was nonetheless a sharp contrast to the lively, sociable scientific community she had been a part of in London. Somerville described how she felt at "a great disadvantage being so entirely deprived of scientific society and of the means of hearing of recent discoveries and new publications" [3]. Both the Royal Institution and the Royal Astronomical Society decreed that Somerville was to be sent copies of their *Proceedings* and the *Greenwich Observations*, respectively, but these did not always materialize and instead she often relied on the goodwill of contemporaries—for example, mathematician Augustus De Morgan—to send her books and papers from Britain.

Beyond communication with her scientific acquaintances via letter, Somerville often hosted visitors in whichever city she was then living. One such visitor was Benjamin Peirce, Professor of Mathematics and Astronomy at Harvard University, USA, who came to Europe in 1870 to view an eclipse. Whilst in Naples he paid Somerville a visit and was so enamoured with his host that, on his return home, he sent her a privately-printed copy of his *Linear Associative Algebra*. In testament to her ongoing reputation, Peirce inscribed the work with: "To the brightest glory of her sex, Mrs Mary Somerville, with the sincere admiration and the profound respect of the Author" [2].

The mathematics in Peirce's *Linear Associative Algebra* would have been entirely new to Somerville. Building on the work of William Rowan Hamilton on quaternions (to which we return later), Peirce considered hyper-complex numbers and presented 162 different algebras in this volume [1, p. 127].Hamilton's quaternions were notable in not satisfying commutativity of multiplication, and Peirce went further in allowing systems in which associativity did not hold, and even those in which division was not well defined.

Apparently struggling to understand these new ideas, which marked a distinctive conceptual shift towards perceiving algebra as the study of structures, Somerville soon began soliciting books to aid her reading of Peirce. In April 1871 she wrote to her publisher, John Murray, telling him of Peirce's book and asking him to send her a copy of Hamilton's work on quaternions. A few months later Hamilton's 1853 *Lectures on Quaternions* were sent to her by the Reverend Whitwell Elwin, a close acquaintance of Murray who was not previously known to Somerville.

It is somewhat unfortunate that Somerville came to the study of quaternions only in 1870, five years after Hamilton's death. The two had met in Cambridge 38 years earlier when Somerville was received at Trinity College, and Hamilton subsequently oversaw her election in 1834 as an honorary member of the Royal Irish Academy. At this time Hamilton was still at the very beginning of his work on quaternions. He was interested in studying complex numbers as 'algebraic couples', or ordered pairs of real numbers on which he defined operations of multiplication and addition. He then began searching for an analogous system of triplets of real numbers, but he struggled to define such a system in which the properties of commutativity, associativity, and distributivity held, and where division (the inverse of multiplication) was well-defined. Eventually, in 1843, Hamilton developed his system of quaternions, namely numbers of the form $a + bi + cj + dk$, where i, j, k are unit vectors such that $i^2 = j^2 = k^2 = -1$. These hypercomplex numbers satisfied all of the properties Hamilton desired, except for commutativity of multiplication; the abandonment of commutativity was revolutionary at the time. In 1848 Hamilton gave a series of four lectures on quaternions at Trinity College Dublin, and these were subsequently expanded into his 1853 book that was sent to Somerville [1, pp. 28–35].

Hamilton's *Lectures* were apparently not sufficient, as Somerville soon reached out to William Spottiswoode, then President of the London Mathematical Society, for further assistance. Like Peirce, Spottiswoode had made Somerville's acquaintance while visiting Naples a few years earlier. In August 1871 Spottiswoode wrote to Somerville:

> I was glad to hear of you again, & especially so as you are still pursuing your studies. As you do not mention the exact subject of the American book, I am not quite sure as to the work best leading to it. But I send you three which I think must cover the ground, & with which I feel sure that you will in any case be interested. . . . The third is [Peter Guthrie] Tait's work on Quaternions. I have sent this rather than Sir William Hamilton's works as the latter are intolerably diffuse, & Tait has carried out the applications of the subject much further than anyone else [7].

During the winter of 1858–59 Tait and Hamilton had shared an intense exchange of letters discussing ideas around quaternions. After Hamilton's death, Tait became one of the leading advocates of quaternions, developing them into a tool with applications in the physical sciences.

A copy of Tait's 1867 *An Elementary Treatise on Quaternions* was part of Somerville's personal scientific library at the time of her death [9]. These books were donated by her daughters as a single collection to Girton College, Cambridge, and the copy of Tait still contains a sheet of handwritten notes by Somerville placed between pages 294 and 295 [10]. This page of formulas hints at the volume of time Somerville spent studying Tait, her last mathematical project.

In the final three years of her life, 1870–72, Somerville was working on numerous publication projects. She revised two mathematical manuscripts that she had first written in the 1830s, prepared new editions of her scientific survey books, and authored her autobiographical *Personal Recollections*. All of these works were left to be published after Somerville's death, for then her government pension would cease, and she intended the royalties from book sales to provide a vital income for her two unmarried daughters.

That Somerville was studying quaternions alongside these authorial projects is clear from manuscript drafts of her *Personal Recollections*. Interspersed with a draft from circa 1872 are sheets on which Somerville has made notes on Peirce's *Linear Associative Algebra*, and some of the draft sheets themselves feature brief jottings of calculations and diagrams; see, for example, Figure 2 [5]. Moreover, in a draft of a letter to Murray from November 1872 Somerville described her morning routine which involved "solv[ing] problems by the higher algebra or add[ing] to the narrative of [her] life" [4].

As can be seen in Figure 3, the draft of this letter was used by Somerville for jotting down calculations and ideas about quaternions. This folio is held with the aforementioned 70 loose sheets, which are a mixture of scrap paper used for rough workings-out and neat pages of notes. Tait provided exercises with no solutions at the end of each chapter of his *Quaternions*, and we can here witness Somerville preparing her own solutions to these problems. Other pages contain summaries of key ideas, and cross-references to results contained in works by Hamilton. One particularly interesting sheet contains an attempt at a proof of a result mentioned but not demonstrated by Tait.

It is highly likely that Somerville was producing these notes for an imagined reader other than herself. As can be seen in the bottom right-hand corner of the sheet in Figure 2, she felt that the quaternion system heralded a new, more powerful age in the mathematical study of the physical sciences. Thus it was a natural continuation of her previous mathematical work advocating for the adoption of analytical methods by those studying the natural world, for instance in her translation of Laplace. In addition, she may have felt that with the recent death of Hamilton, and Tait's declaration that he had moved on from his studies of quaternions, there was a lucrative gap in the book market that she could profit from and aid her daughters. The archival materials provide

further textual evidence that Somerville was in the early stages of preparing a book, possibly a companion to Tait's own. She wrote up multiple copies of the same sheets, returned to others to make edits, and at the beginning of her notes on Chapter 3 she declared that "the whole of this chapter is difficult and requires explanation which I have attempted" [4]. This justification for the material that she has produced and the neatly copied-up pages of notes would be unnecessary, were Somerville writing only as part of her own mathematical learning practice.

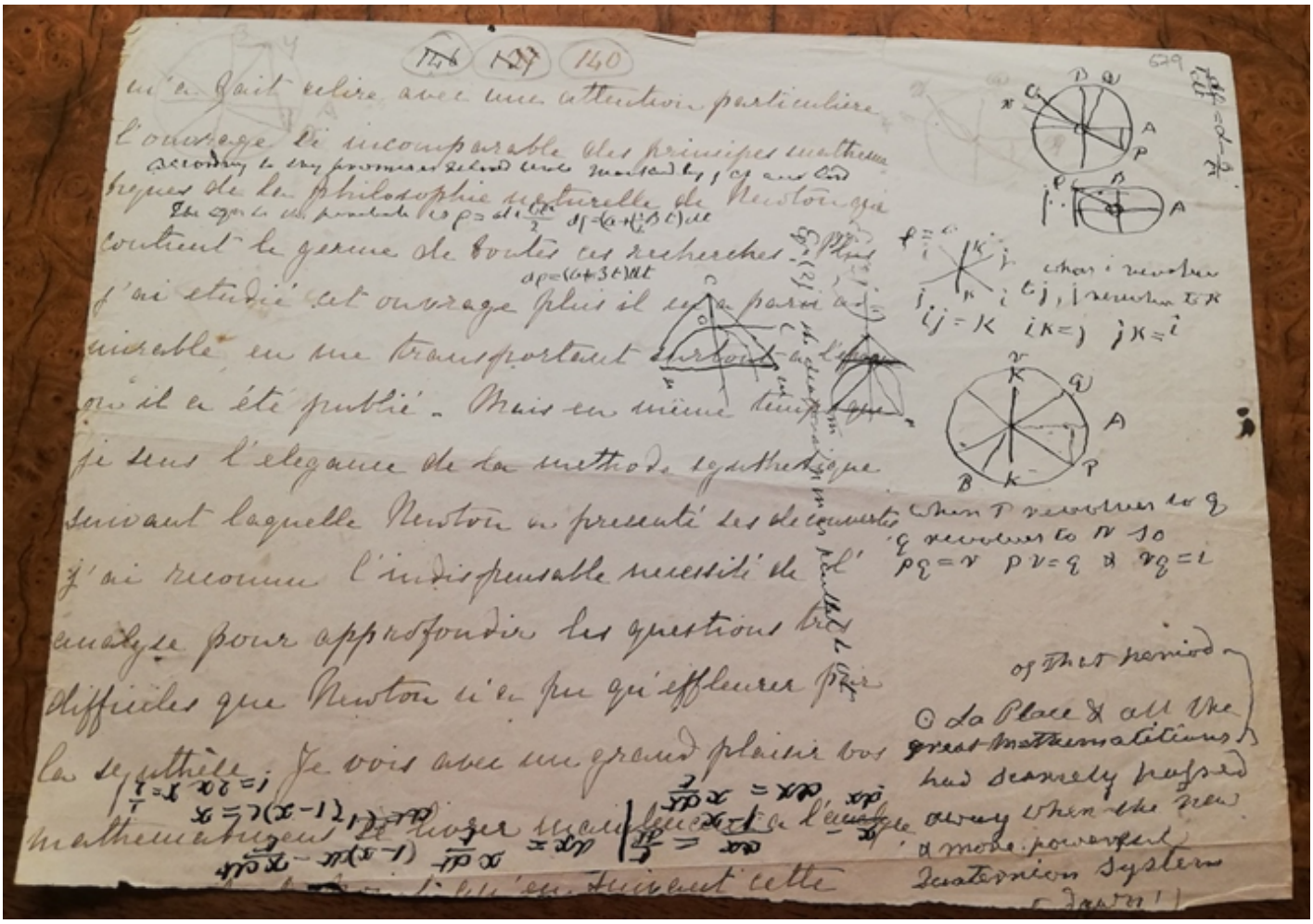

Figure 2. Image of a sheet held with a draft of Somerville's Personal Recollections. This page features a letter from Laplace copied up in Somerville's hand, which was included in the final publication [6, p.181]. Reproduced with the kind permission of the Principal and Fellows of Somerville College, Oxford.

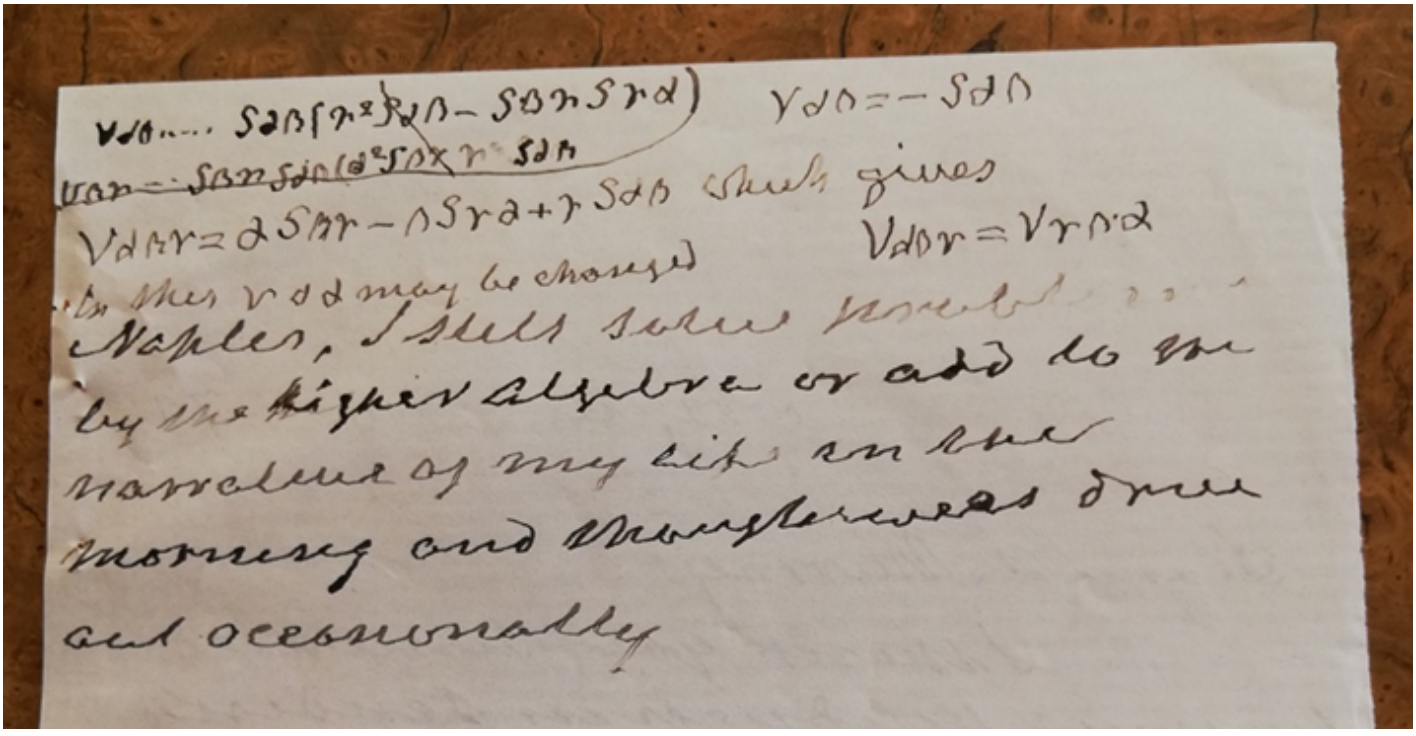

Figure 3. Draft of letter written to publisher John Murray, dated November 1872. Reproduced with the kind permission of the Principal and Fellows of Somerville College, Oxford.

Unfortunately, Somerville's work on the quaternions never came to fruition. She continued studying Tait's book until the day she died, November 29, 1872, at which point she had only written notes up to Chapter 3 [6, p. 376]. Her two other mathematical manuscripts were also left unpublished, but her autobiography was well received and two of her scientific survey books were re-issued after her death, providing some income for her daughters.

These loose sheets full of letter drafts, notes, and calculations raise questions about opportunities for publishing mathematical books in nineteenth-century Britain. Throughout her life Somerville was described as a mathematician, and she described mathematics as the subject she found most congenial. Yet her career as an author was focused on books which detailed results in the physical sciences, without delving into the mathematics used to reach them. It is impossible to say for certain that Somerville was preparing a work on quaternions for publication, but it is clear that at the end of her life she again hoped to reconcile her need for financial stability and her own intellectual gratification through mathematical writing.

Somerville's exposure to emerging ideas in algebra was contingent on the respect and recognition she commanded in scientific circles, which led men such as Peirce and Spottiswoode not only to visit her, but to provide her with recent mathematical works. When she faced difficulties, she did what she had done for over sixty years and turned to her acquaintances to ask for help. It is a testament to her tenacity that rather than being dismayed and discouraged by her difficulties in understanding the works of Peirce, Tait, and Hamilton, Somerville instead enjoyed the opportunity to study an entirely new branch of mathematics. When writing about solving the exercises in Tait, she admitted:

> Sometimes I find them difficult, but my old obstinacy remains, for if I do not succeed to-day, I attack them again on the morrow [6, p. 364].

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[10] Thanks to Jenny Blackhurst, Librarian and Fellow at Girton College, Cambridge, for consulting this material on my behalf.

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