

Recurring decimals, proof, and ice floes

Education Notes

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Education Notes bring mathematical and educational ideas forth to the CMS readership in a manner that promotes discussion of relevant topics including research, activities, issues, and noteworthy news items. Comments, suggestions, and submissions are welcome.

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Or: Why do we teach students how to prove things we all know already, such as $0.999\ldots = 1$?

Partly, of course, so that they develop thinking skills to use on questions whose truth-status they won't know in advance. Another part, however, concerns the dialogue nature of proof: a proof must be not only correct, but also persuasive: and persuasiveness is not objective and absolute, it's a two-body problem. Not only to tango does one need two.



The statements —

(1) *ice floats on water,*

(2) *ice is less dense than water*

— are widely acknowledged as facts and, usually, as interchangeable facts. But although rooted in everyday experience, they are not that experience. We have firstly represented stuffs of experience by sounds English speakers use to stand for them, then represented these sounds by word-processor symbols that, by common agreement, stand for them. Two steps away from reality already! This is what humans do: we invent symbols for perceived realities and, eventually, evolve procedures for manipulating them in ways that mirror how their real-world origins behave. Virtually no communication between two persons, and possibly not much internal dialogue within one mind, can proceed without this. Man is a symbol-using animal.

Statement (1) counts as fact because folk living in cooler climates have directly observed it throughout history (and because conflicting evidence is lacking). Statement (2) is factual in a significantly different sense, arising by

further abstraction from (1) and from a million similar experiential observations: partly to explain (1) and its many cousins, we have conceived ideas like mass, volume, ratio of mass to volume, and explored for generations towards the conclusion that mass-to-volume works out the same for similar materials under similar conditions, and that the comparison of mass-to-volume ratios predicts which materials will float upon others.

Statement (3): 19 is a prime number. In what sense is this a fact? Its roots are deep in direct experience: the hunter-gatherer wishing to share nineteen apples equally with his two brothers or his three sons or his five children must have discovered that he couldn't, without extending his circle of acquaintance so far that each got only one, long before he had a name for what we call 'nineteen'. But (3) is many steps away from the experience where it is grounded. It involves conceptualisation of numerical measurements of sets one encounters, and millennia of thought to acquire symbols for these and codify procedures for manipulating them in ways that mirror how reality functions. We've done this so successfully that it's easy to forget how far from the tangibles of experience they stand.

Statement (4): $\sqrt{2}$ is not exactly the ratio of two whole numbers. Most first-year mathematics students know this. But by this stage of abstraction, separating its fact-ness from its demonstration is impossible: the property of being exactly a fraction is not detectable by physical experience. It is a property of how we abstracted and systematised the numbers that proved useful in modelling

reality, not of our hands-on experience of reality. The reason we regard $\sqrt{2}$'s irrationality as factual is precisely because we can give a demonstration within an accepted logical framework.

What then about recurring decimals? For persuasive argument, we must first ascertain the distance from reality at which the question arises: not, in this case, the rarified atmosphere of undergraduate mathematics but the primary school classroom. Once a child has learned rituals for dividing whole numbers and the convenience of decimal notation, she will try to divide, say, 2 by 3 and will hit a problem: the decimal representation of the answer does not cease to spew out digits of lesser and lesser significance no matter how long she keeps turning the handle.

What should we reply when she asks whether zero point infinitely many 6's is or is not two thirds, or even — as a thoughtful child should — whether zero point infinitely many 6's is a legitimate symbol at all?

The answer must be tailored to the questioner's needs, but the natural way forward — though it took us centuries to make it logically watertight — is the nineteenth-century definition of sum of an infinite series. For the primary school kid it may suffice to say that, by writing down enough 6's, we'd get as close to $2/3$ as we'd need for any practical purpose. For differential calculus we'd need something better, and for model-theoretic discourse involving infinitesimals something better again. Yet the underpinning mathematics for equalities like $0.6666\ldots = 2/3$ *where the question arises* is the nineteenth-century one. Its fact-ness therefore resembles that of ice being less dense than water, of 19 being prime or of $\sqrt{2}$ being irrational: it can be demonstrated within a logical framework that systematises our observations of real-world experiences. So it is a fact not about reality but about the models we build to explain reality. Demonstration is the only tool available for establishing its truth.

Mathematics without proof is not like an omelette without salt and pepper; it is like an omelette without egg.

About the Authors



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Aisling McCluskey and Brian McMaster have co-authored several textbooks in the areas of topology, analysis and complex numbers.