Teaching Calculus Through History’s Lens

CMS NOTES

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Try this. Go to your library and, for each decade after 1950, take down a typical Calculus textbook first published during those ten years. You will find that, by and large, most of the textbooks used to teach Differential Calculus in college classrooms for the past 70 or 80 years have presented the topics in essentially the following order. (The dates below indicate when each concept was introduced. Obviously, these should be taken with a grain of salt. New concepts tend to emerge slowly rather than via a single invention.)

- Limits (1823)
- Continuity (1817)
- The Derivative (1797)
- Differentiation Rules (1684) and Tangent Lines
- Implicit Differentiation (1684)
- Related Rates (1665)
- Rolle’s Theorem (1691), the Mean Value Theorem (1797), and Extreme Value Theorem (1874)
- The First Derivative Test and Curve Sketching
- The Second Derivative Test and Curve Sketching
- Applied Optimization
- Antiderivatives

Logically, this makes perfect sense. We start with the theoretical foundation and build from there. But the differentials on which Leibniz founded his Calculus were used for nearly 200 years before limits were invented to replace them. Notice that differentials don’t appear in our list. They sometimes make a small cameo appearance in a section on approximations, but they are otherwise absent from standard teaching practices.

Note also that, as we read this list from top to bottom, we move backwards in time. This is common in mathematics teaching. In order to present a “clean” finished product to students we start with the foundational underpinnings and then move on to the motivating applications.

Think about your own research. When wrestling with a research problem are the foundational questions top-of-mind, or do you keep those on the back burner until you’ve actually made progress? Learning new mathematics is a creative process, whether one is doing original research or learning in a classroom. But when we teach in reverse-historical order, we are illustrating neither how mathematics is best created nor how it is best learned. We are illustrating how it is best presented. We (the authors) believe that preserving the historical order helps students see and appreciate the creative, problem-solving aspect of our discipline better than the reverse-historical order that has been the norm for some decades.

Replacing limits with differentials as the basis for Calculus is entirely within reach of first-year students if we are willing to proceed intuitively. Indeed, Leibniz based his “Calculus Differentialis” (Calculus of Differentials) on the notion of a differential because it is both intuitive and visually suggestive. He knew it was not a strong logical foundation, but his goals were to construct tangents and to solve optimization problems in a systematic way—his Calculus was a means to those ends. This is manifest from the title of his first publication on the topic:

A New Method for Maxima and Minima as Well as Tangents, Which is Impeded Neither by Fractional Nor by Irrational Quantities, and a Remarkable Type of Calculus for This [1].

On the other hand, the limit concept was suggested by Newton, refined by Cauchy (and others), and finalized by Weierstrass in 1874, nearly 200 years after Leibniz. Like all of mathematics, the limit concept was invented to address very specific questions—in this case, how can we be sure that the computational procedures described by Leibniz (and Newton) really work? Perhaps more important, when do they fail? When we begin a Calculus course with the non-intuitive subtlety of the limit concept, we are asking students to understand the answers to those questions without actually asking the questions first, or even indicating why they are important. Although this may be good mathematics, it seems to us to be poor pedagogy. It is like teaching a student to solve a crossword puzzle by giving them the grid, the answers, and their locations, and telling them to fill in the grid. Certainly, something will be learned this way. And, we can build from there. But how many students would stick with us until it gets interesting? Would you?

Rather than starting at the logical beginning, suppose we start at the chronological beginning, with Leibniz' Calculus of Differentials. The differentiation rules are easy to explain intuitively using differentials and diagrams. As Leibniz stated in his paper, "The Demonstration of all this [the differentiation rules] will be easy to one who is experienced in these matters [infinitesimals]" [1]. A beginning Calculus student might balk at calling them "easy," but once differentials are accepted, the differentiation rules are straightforward. In fact, the Constant Rule $(d(constant) = 0)$ and the Sum Rule $(d(x + y) = dx + dy)$ are directly analogous to their finite counterparts. The subtleties of differentials will force a later switch to limits, but having students accept these objects in the beginning is akin to having beginning geometry students accept that there is a geometric object called a point.

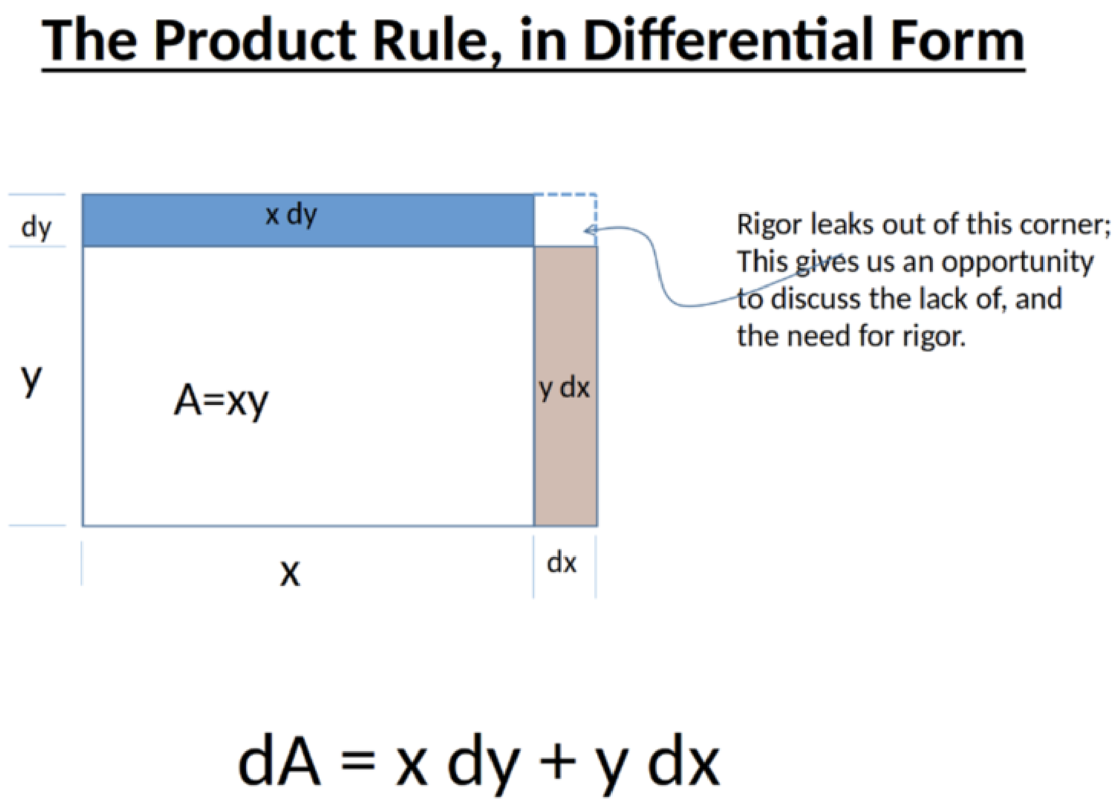

Figure 1. Diagrammatic proof of the Product Rule. Created by the authors.

Figure 1 shows a simple proof of the Product Rule. In order to believe that the Product Rule is valid, a student only needs to accept that the corner indicated can be safely ignored. This is not a rigorous proof. But it is intuitive and therefore believable. Moreover, the lack of rigor can be pointed out in passing, thereby sowing the seeds of curiosity: "I wonder what a rigorous proof would look like?"

We contend that a clear, but intuitive, explanation will give the beginner more faith in a computational tool than any fully rigorous, but abstruse proof. All that is required to use a tool effectively is faith that it is the right tool for the job—and practice. Lots of practice.

Once these first three differentiation rules are known, the Constant Multiple Rule, the Power Rule, and the Quotient Rule all follow without much fuss (assuming that functions involved are differentiable). They can all be developed, and substantial practice given, fairly quickly.

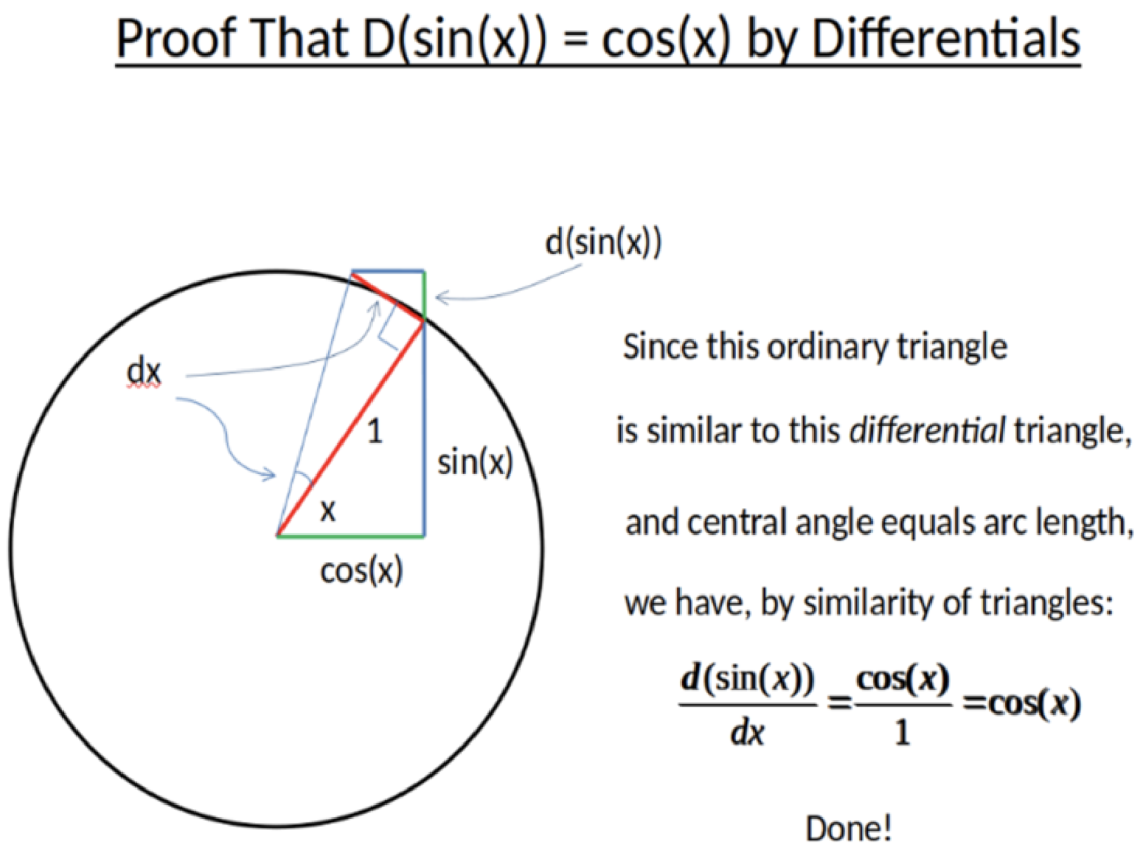

Figure 2. Proof by differentials of $\frac{d(sin(x))}{dx} = cos(x)$ Diagram created by the authors.

Differentiating the trigonometric functions without limits is similarly straightforward. For example, Figure 2 shows a differential-based proof that the derivative of sin x is cos x. (This proof was devised by Roger Cotes, a colleague of Newton.)

Again, nothing here is meant to be rigorous. We simply provide some *ad hoc*, intuitive arguments that can be used to justify the differentiation rules without limits so that we can quickly move on to using the computational tools of Calculus.

For beginning students, the use of function notation and Lagrange's prime notation— $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$—can make the differentiation of composed functions unnecessarily unwieldy. But using differentials renders this moot. For example, if we let $z = \sin x$ then

$$d(\sin^2 x) = d(z^2) = 2zdz = 2(\sin x)d(\sin x) = 2\sin x \cos x dx.$$

When we use differentials, topics such as implicit differentiation, the chain rule, and related rates do not warrant special consideration. For example, given an equation such as $x^2 + y^2 = 1$, direct application of the differentiation rules yields $2xdx + 2ydy = 0$. If we wish to compute a slope, we divide by dx to obtain $\frac{dy}{dx} = \frac{-x}{y}$. If we wish to see how the rates of change of x and y are related, we divide by dt to obtain $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$ or, in Newton's fluxion notation, $2x\dot{x} + 2y\dot{y} = 0$. There is no need to "remember the chain rule" because the chain rule did not exist in either Newton's or Leibniz' version of Calculus.

In fact, the phrase "chain rule" doesn't seem to appear in Calculus texts until the late 19th or early 20th centuries, although earlier arithmetic and algebra books used the term to describe the computations involved in changing units (e.g. $\frac{feet}{second} = \frac{mile}{hour} \cdot \frac{feet}{mile} \cdot \frac{hour}{minute} \cdot \frac{minute}{second}$), which is very similar to the Calculus chain rule expressed in differential form. We conjecture that textbook authors simply co-opted the name of the older rule for use in Calculus.

Computations of this kind are similar to, and are therefore good preparation for, the types of calculations students will be expected to perform later in contexts where differentials are already heavily relied upon. Think of integration by substitution, integration by parts, and line and path integrals.

Furthermore, this approach is not constrained to single-variable functions. For example, if $z = x^2y^3$ then

$$dz = x^2(3y^2dy) + y^3(2xdx) = 2xy^3dx + 3x^2y^2dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy.$$

Notice that the partial derivatives emerged quite naturally.

Freedom from the constraint of one-variable Calculus can be a powerful tool. Consider the derivation of Snell's Law of Refraction using Figure 3.

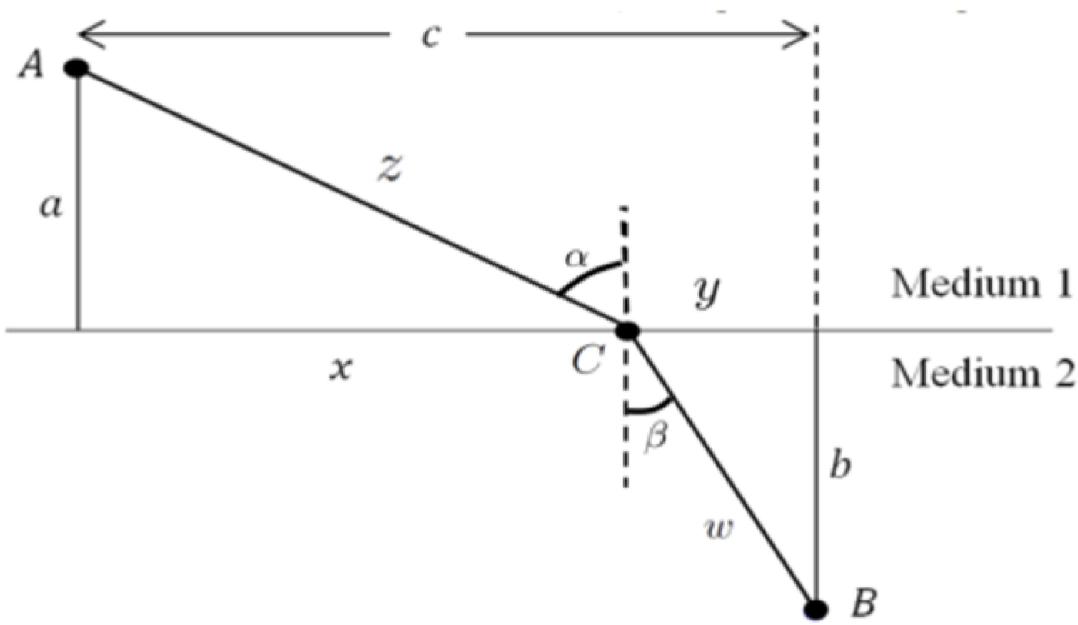

Figure 3. Visualization of the derivation of Snell's Law of Refraction. Diagram created by the authors.

Let v_1 and v_2 represent the velocities of light in medium 1 and medium 2, respectively. The total time for light to travel from point A to point B is given in terms of the single variable x by

$$T(x) = \frac{\sqrt{a^2+x^2}}{v_1} + \frac{\sqrt{b^2+(c-x)^2}}{v_2}.$$

Minimizing $T(x)$ to find the path that light travels is messy, and it only gives us the value of x that minimizes $T(x)$. To derive Snell's Law we still have to show that $\frac{\sin\alpha}{v_1} = \frac{\sin\beta}{v_2}$. If instead we write down our objective function

$$T = \frac{z}{v_1} + \frac{w}{v_2}$$

along with all of our constraints

$$x + y = c, \qquad z^2 = x^2 + a^2, \qquad w^2 = y^2 + b^2$$

in terms of the variables that appear in the diagram and then differentiate (using differentials) we get

$$dx + dy = 0, \qquad 2zdz = 2xdx, \qquad 2wdw = 2ydy$$

so that

$$dT = \frac{dz}{v_1} + \frac{dw}{v_2} = \frac{1}{v_1} \cdot \frac{x}{z} dx + \frac{1}{v_2} \cdot \frac{y}{w} dy = \frac{1}{v_1} \cdot \frac{x}{z} dx - \frac{1}{v_2} \cdot \frac{y}{w} dx.$$

Setting $\frac{dT}{dx} = 0$ gives us $\frac{\left(\frac{x}{z}\right)}{v_1} = \frac{\left(\frac{y}{w}\right)}{v_2}$ or $\frac{\sin \alpha}{v_1} = \frac{\sin \beta}{v_2}$, which is Snell's Law.

Treating $\frac{dT}{dx}$ as a ratio of two differentials rather than as a single entity can streamline the use of Calculus as a problem-solving tool in many instances.

You may not yet be fully convinced that the "differentials first" approach is better pedagogy, but if you are still reading you are clearly intrigued and you are probably wondering where you could find a textbook that takes this approach.

We're so glad you asked. We have written such a book, and we have used it ourselves in the classroom. Our text, *Differential Calculus: From Practice to Theory*, is an Open Educational Resource (OER) that you are welcome to download from https://milneopentextbooks.org/differential-calculus-from-practice-to-theory/ and use in your Differential Calculus course at no cost to you or your students. It has a Creative Commons license, which means that you are welcome to use it as is or to alter it to suit your needs. We deliberately designed our textbook so that a student learning from it will end the first semester with at least the same skills and understanding as a student learning from a "limits first" approach.

Part I of our textbook (*From Practice*) presents "Calculus Differentialis" as the intuitive, focused-on-problem-solving "Remarkable Calculus" that Leibniz described in his paper [1]. The differentiation rules are introduced intuitively, leaning heavily on the notion of the differential. We do not hide the questionable nature of the differential; we just don't dwell on it. When the use of differentials becomes problematic we point that out so the student is aware of the issue and, hopefully, becomes curious. But we defer the resolution of this foundational question to Part II (*to Theory*), where we develop the limit concept with full epsilon/delta rigor.

This is a theme throughout our textbook. Substantial questions (e.g., "What is the shape of a hanging chain?") are often introduced before they can be easily answered. The students are then led to partial solutions. Once the necessary theory and techniques have been developed the problem is then re-addressed. The goal is to induce curiosity, and to provoke questions in the student ("If these differentials aren't really a viable foundation, what is?")—and thus reproduce the research experience as much as possible in the context of a Calculus classroom.

Reference

[1] Leibniz, Gottfried Wilhelm. (1684, Oct.) Nova methodus pro maximis et minimis, itemque tangentibus, quae nec fractas, nec irrationales quantitates moratur, et singular pro illis calcului genus. *Acta Eruditorum*, 467–473.

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