CMS NOTES

Geometric Transformations, 1800–1855

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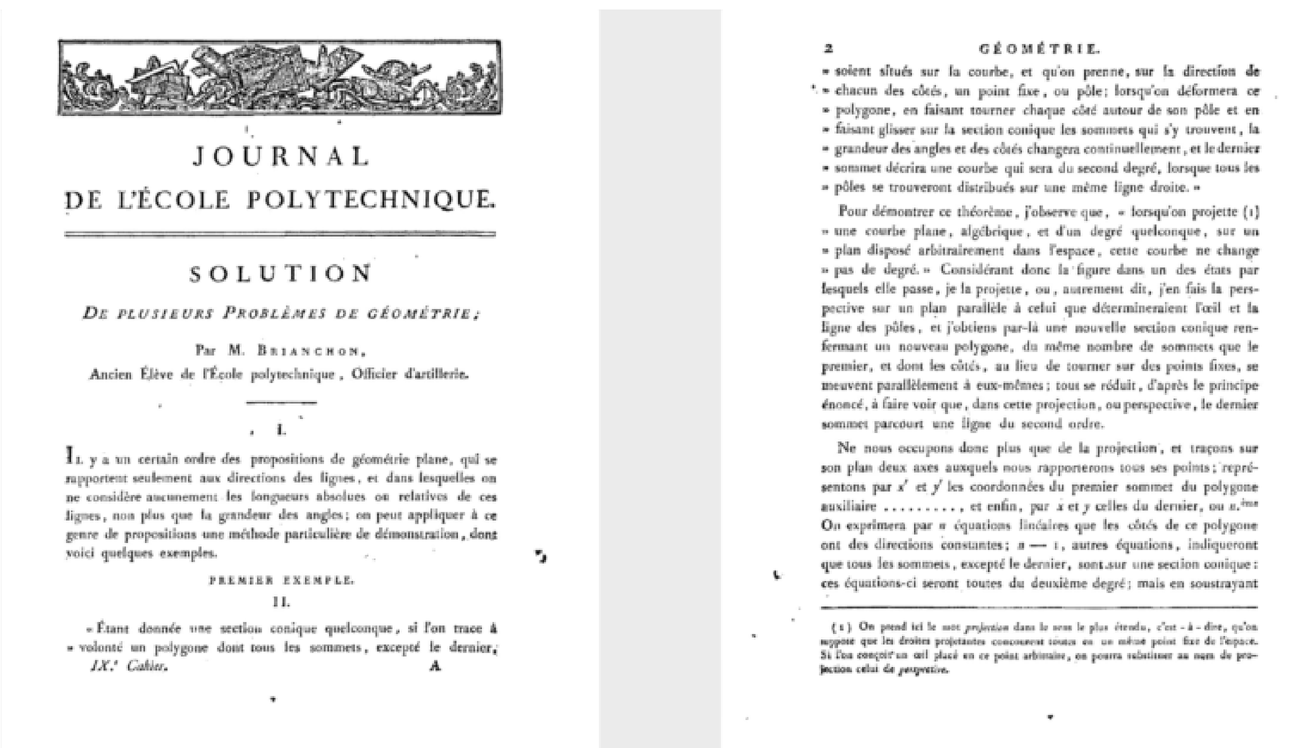
The nature of geometry changed dramatically between 1800 and 1855. My interest in that topic and time period began a number of years ago, as my enthusiasm for geometry, particularly of the synthetic type, merged with an even older enthusiasm for history. Following geometry through the centuries, I saw an interesting story line emerging. Ancient Greeks envisioned geometric figures as stationary, and they showed almost no interest in propositions about collinear points or concurrent lines. The exceptions are Menelaus's Theorem and various propositions from Pappus, but those ideas emerged only in the late ancient period.

Coherent work involving movement of a sort, by plane-to-plane projection, together with incipient projective geometry and propositions about collinear points and concurrent lines, appeared in a burst in the 17th century, with Girard Desargues [6; 7], Blaise Pascal [15], and Philippe de La Hire [9]. (It is worth noting that Desargues made great use of Menelaus's Theorem.) But, for lack of interest, the publications by these authors that were the most projective in approach were soon lost or forgotten.

Around 1800, the atmosphere changed. After the publication of Adrien-Marie Legendre's *Éléments de géométrie* of 1794 [10], which would have felt familiar to Greek geometers of two millennia earlier, Gaspard Monge [14] and Lazare Carnot [4; 5] presented a variety of propositions involving collinear points and concurrent lines. Forgotten propositions, including Desargues's Theorem and Pascal's Hexagon Theorem, were rediscovered. They appeared at first without credit to their discoverers, and proofs were not projective, but in 1822 J. V. Poncelet [20] would prove them by projective methods and properly note their discoverers. Of particular importance to Poncelet was an 1810 article by Charles Julien Brianchon [3] in which he solved several problems by a plane-to-plane projection that let one line of points "pass to infinity," as Poncelet put it.


JOURNAL

DE L'ÉCOLE POLYTECHNIQUE.

SOLUTION

De plusieurs Problèmes de géométrie;

Par M. Brianchon,
Ancien Élève de l'École polytechnique, Officier d'artillerie.

I.

Il y a un certain ordre des propositions de géométrie plane, qui se rapportent seulement aux directions des lignes, et dans lesquelles on ne considère aucunement les longueurs absolues ou relatives de ces lignes, non plus que la grandeur des angles; on peut appliquer à ce genre de propositions une méthode particulière de démonstration, dont voici quelques exemples.

PREMIER EXEMPLE.

II.

« Étant donnée une section conique quelconque, si l'on trace à » volonté un polygone dont tous les sommets, excepté le dernier,

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» soient situés sur la courbe, et qu'on prenne, sur la direction de » chacun des côtés, un point fixe, ou pôle; lorsqu'on déformera ce » polygone, en faisant tourner chaque côté autour de son pôle et en » faisant glisser sur la section conique les sommets qui s'y trouvent, la » grandeur des angles et des côtés changera continuellement, et le dernier » sommet décrira une courbe qui sera du second degré, lorsque tous les » pôles se trouveront distribués sur une même ligne droite. »

Pour démontrer ce théorème, j'observe que, « lorsqu'on projette (1) » une courbe plane, algébrique, et d'un degré quelconque, sur un » plan disposé arbitrairement dans l'espace, cette courbe ne change » pas de degré. » Considérant donc la figure dans un des états par lesquels elle passe, je la projette, ou, autrement dit, j'en fais la perspective sur un plan parallèle à celui que détermineraient l'œil et la ligne des pôles, et j'obtiens par-là une nouvelle section conique renfermant un nouveau polygone, du même nombre de sommets que le premier, et dont les côtés, au lieu de tourner sur des points fixes, se meuvent parallèlement à eux-mêmes; tout se réduit, d'après le principe énoncé, à faire voir que, dans cette projection, ou perspective, le dernier sommet parcourt une ligne du second ordre.

Ne nous occupons donc plus que de la projection, et traçons sur son plan deux axes auxquels nous rapporterons tous ses points; représentons par x' et y' les coordonnées du premier sommet du polygone auxiliaire ...., et enfin, par x et y celles du dernier, ou n.ième On exprimera par n équations linéaires que les côtés de ce polygone ont des directions constantes; $n - 1$, autres équations, indiqueront que tous les sommets, excepté le dernier, sont sur une section conique: ces équations-ci seront toutes du deuxième degré; mais en soustrayant

(1) On prend ici le mot *projection* dans le sens le plus étendu, c'est-à-dire, qu'on suppose que les droites projetantes concourent toutes en un même point fixe de l'espace. Si l'on conçoit un œil placé en ce point arbitraire, on pourra substituer au nom de projection celui de *perspective*.

Figure 1. The first two pages of Brianchon's article [3]. Gallica.

This began a flood of geometric transformations and relationships, called *Verwandtschaften* by A. F. Moebius in his *Der Barycentrische Calcul* [11]. That 1827 work gave the first overview of transformations. Moebius examined four different geometric relationships, starting with *equality*, *similitude*, and *affinity*. *Similitude*, now called *dilation*, the topic of Euler's E693 [8], was thoroughly developed by Poncelet in 1813, 1820, and 1822 [18; 19; 20]. Moebius's fourth transformation was new and the most general, the *collineation*. He set his transformations in a plane whose elements were real triples, except $(0,0,0)$, with the homogeneity property, namely, that for $k \neq 0$ triples (a,b,c) and (ka,kb,kc) represent the same point. Moebius demonstrated a remarkable property, that given any four points A, B, C, D, no three collinear, then any point P could be represented as $aA + bB + cC + dD$, and for any choice of four points A', B', C', D', no three collinear, then $P' = aA' + bB' + cC' + dD'$ defines a collineation, and any collineation could be defined this way. (He characterized an *affinity*, a collineation mapping parallel lines to parallel lines, in a corresponding way, with three non-collinear finite points.) With modification to Moebius's homogeneous coordinate system by Julius Plücker in 1831 [17], the triples form what we now recognize as the *real projective plane*. In that plane, a *collineation* is the same as a *projectivity*.

Plane-to-plane projection was the basis for *homology*, as developed by Poncelet in his 1822 *Traité des Propriétés Projectives des Figures* [20]. The line of intersection of the two planes, the *axis*, is a line of fixed points. Poncelet showed that a homology can operate in a single plane when one of the two planes is rotated about the axis to coincide with the other, but still with a line of fixed points.

The *projective transformation*, in synthetic geometry, was created by Jacob Steiner [23]. He began with *elementary forms*, in particular the *line* of points and the *pencil*, *Strahlbüschel*, of concurrent lines; then related the elementary forms by *perspectivities*; and, finally, defined a *projectivity* as a composition of perspectivities.

Where Poncelet, in 1813, had declared that a conic section was the plane-to-plane projection of a circle, Steiner defined a point conic to be the set of points of intersection of corresponding lines in projectively related pencils. (Poncelet's *Notebooks*, written as a prisoner in Russia in 1813–1814, were only published in 1862 [21].)

Another transformation, *inversion* (over a circle), anticipated in work of the 1820s, was developed into modern form by Giusto Bellavitis in 1836 [2].

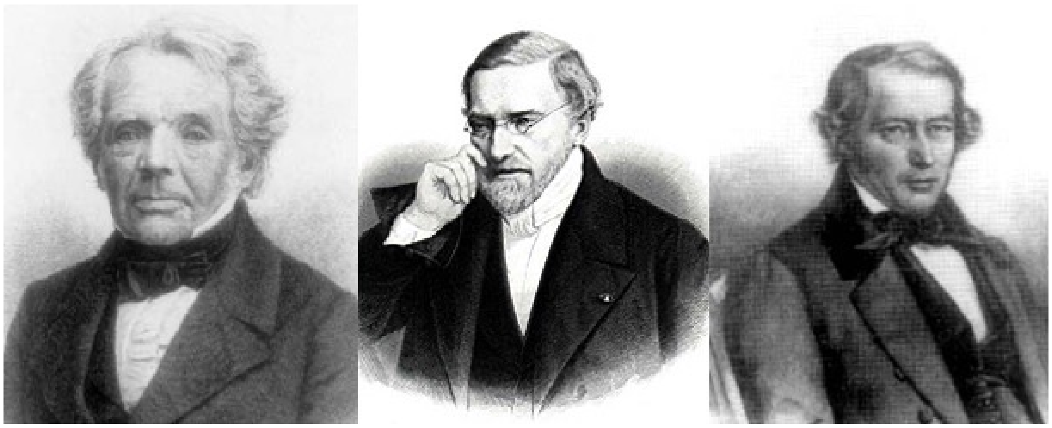

Figure 2. Moebius, Poncelet, and Steiner. *Convergence* Portrait Gallery.

Why do I end this period of transformations in 1855? First, I wished to include important works in the line of development sketched above, especially K. G. C. von Staudt's 1847 *Geometrie der Lage* [24]. Von Staudt placed projective geometry in a more abstract setting; his book, for example, has no figures. The book includes two proofs of Moebius's theorem that a projectivity is a collineation determined by four points, no three collinear, and their images. Von Staudt committed himself to removing any dependence on length from his geometry. Significantly, the cross-ratio of four collinear points, whose invariance was Steiner's main identifying property of a projective relation, depends on length, so von Staudt chose an alternative, invariance of the harmonic relation. That works because the *complete quadrilateral*, introduced by Carnot, was a non-metric concept, and it determined a harmonic set of four points. In later years, geometers tended to follow von Staudt's penchant for abstraction, and many would require that a truly projective geometry be non-metric.

A second important transitional work was Moebius's second paper [12] on what has come to be called the *Moebius transformation*. Without using the function definition of the *Kreisverwandtschaft*, as he called it, Moebius derived the familiar properties, including the invariance of the cross-ratio among four complex points, not necessarily collinear. He was a pioneer in setting his work in the complex plane.

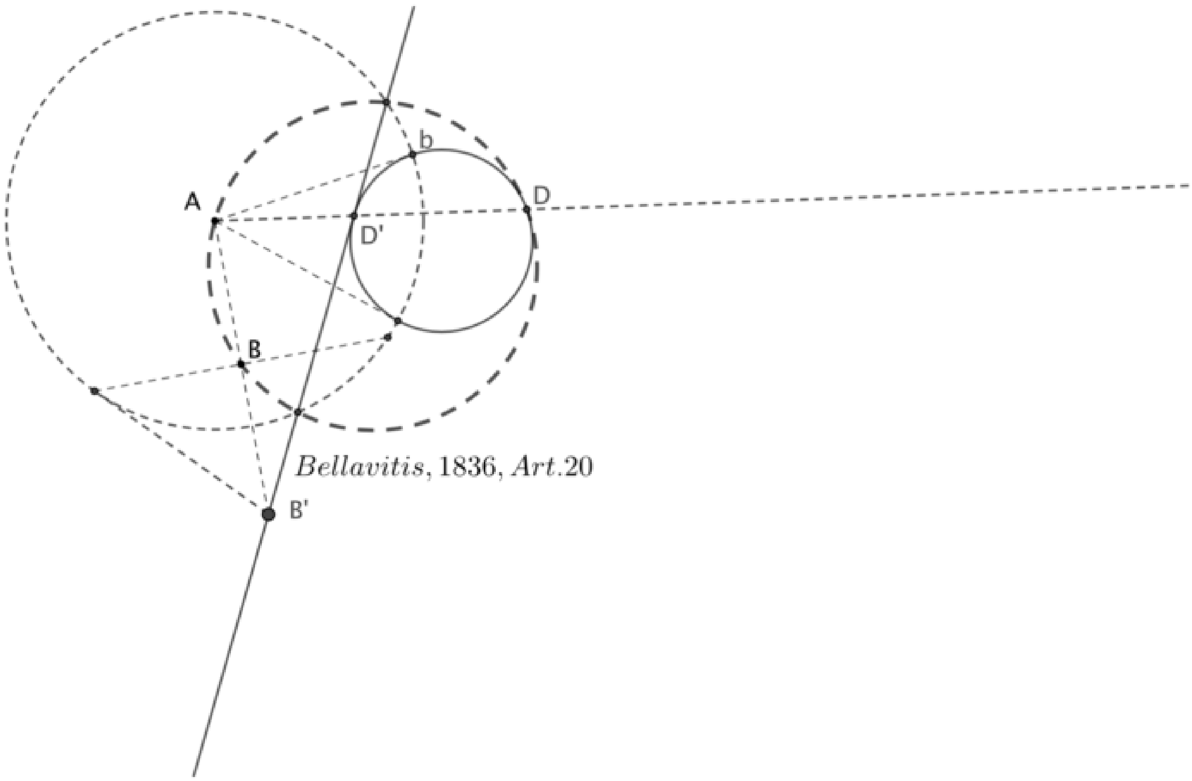

Figure 3. Illustration of inversion described by G. Bellavitis [1836, Art. 20], "Construct a circle which passes through two given points $A \backslash B$ and is tangent to a given circle Db." Figure created by the author.

Two notable works signaled a new direction in geometry. The first was Bernhard Riemann's inaugural lecture at Göttingen in 1854, on the bases of geometry [22]. The second involved von Staudt. While his 1847 *Geometrie der Lage* followed the trajectory of contemporary work in projective geometry, by contrast his *Beiträge zur Geometrie der Lage*, which appeared in three volumes from 1856 to 1859 [25], introduced a complex projective space and an algebra derived from geometric axioms, and it greatly influenced geometry in the coming years.

And where does my work with this material stand? A narrowly focused paper, "Poncelet's discovery of homology," was recently published [1]. My current project is a book that combines history with the exposition of the mathematics involved, written with an undergraduate mathematics major in mind. It is the sort of project that retirement makes possible.

Christopher Baltus earned a PhD at the University of Colorado in 1984, and taught 33 years at SUNY Oswego. He and his wife retired in 2019 and moved to Poughkeepsie, NY. In retirement he still rides a bicycle, reads history and mathematics, and is a volunteer mathematics aide in an elementary school. He notes that this article was written under guidance from Hardy Grant, whose wisdom and kindness will be greatly missed.

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