Kseniya Garaschuk (University of the Fraser Valley)

Education Notes bring mathematical and educational ideas forth to the CMS readership in a manner that promotes discussion of relevant topics including research, activities, issues, and noteworthy news items. Comments, suggestions, and submissions are welcome.

John Grant McLoughlin, *University of New Brunswick (johngm@unb.ca)*
Kseniya Garaschuk, *University of the Fraser Valley (kseniya.garaschuk@ufv.ca)*

> "We must remember that intelligence is not enough. Intelligence plus character – that is the goal of true education. The complete education gives one not only power of concentration, but worthy objectives upon which to concentrate."
>
> **Martin Luther King**

Real-life, realistic, relevant, genuine, authentic — we use these words nearly interchangeably to describe desirable properties of application problems we would like to see in math courses. Specifically, problems we would like to utilize in first-year service courses (e.g. life sciences streams) and highly specialized ones (e.g. engineering). Yet, not only do we struggle to pick a word (I will stick with authentic), we often struggle to describe exactly what properties such applications should possess. We certainly know what we do not want: we don't want chicken coops, sliding ladders, boats pulling away from the dock, people walking away from a street lamp, lighthouses shining light on the shore. While having non-examples is useful, it would be more useful to have guidelines to help identify where the authenticity fails and where it is created.

Presented here are some key features that I find valuable in defining authentic applications, based on my teaching experience. It's important to note that this list is not exhaustive, and these features may overlap, describing similar concepts from different perspectives — essentially, these are my field notes derived from practical classroom experience.

Real situations, information and data

The "application" in standard application problems refers to the ability to utilize learnt mathematical techniques rather than the relationship between the problem's context and the real world. As in, the problem calls for you to *apply* your skills, not that the problem itself is *applied*. The context serves only as a light dusting of makeup on top of familiar processes. These problems fit perfectly on the appropriately named "Apply" level of Bloom's taxonomy: use information in a new (but similar) situation (see endnote 1).

Authentic problems need to be rooted in real situations, real information available or obtainable by someone working on this problem and the real data that comes with it. Real scenarios tend to be complex and real data is messy, so this presents a perfect opportunity to have serious discussions about simplifying assumptions. What part of the problem will we focus on, what aspects of it will we drop, what assumptions will we declare to be able to approach this problem, how will these assumptions affect our findings?… This reflective process mirrors how professionals of various fields employ mathematical problem solving in the elusive "real world" and students deserve to see this power of math early on in their learning journey.

Scenarios that students are aware of outside of classroom

Relevance is an important aspect of motivating learning. But "relevance" itself is a loaded concept. My students are half my age, they come from different cultural/educational/linguistic/… backgrounds than I do, they have diverse career plans, diverse values and beliefs. How can I possibly know what is relevant to them?

First of all, ask. Or, as per Gloria Steinem, "Always ask the turtle." (see endnote 2). Students rarely get asked what excites them, what they value, what is relevant and what they *actually* want to learn about. So let's start asking. Secondly, do not assume that because you are teaching a class of biology majors that all your students will be fascinated by all things biological. There are many facets to biology itself: I doubt that every single biology major finds cell growth riveting in a same way that I know that not every single math major thinks that solving trigonometric equations is the bee's knees. We are all complex individuals with multiple interests, so students majoring in subject X should not be put in a box labeled X and presented with nothing but applications relevant to X.

I ask my students for their input in a variety of ways, but here are three of them that I have found most effective in soliciting ideas or even full-fledged usable applications. One-pagers are a quick way for students to synthesize and present a high-level view of a concept of their choice together with an example of an application. Bi-weekly learning journal submissions often result in students suggesting what material from other courses or from their non-academic lives they would like to see me incorporate into the course – everything from lake pollution to toilet paper shortage at the beginning of the pandemic. But my current favourite assignment comes (sadly) at the very end of the term: I ask my students to design a long-answer final exam question. I give directions guiding the students in the creation of questions that are meaningful and focus on the application of knowledge, but I explicitly allow lots of room for creativity. Last year, across two sections of a course with 70 students total, this assignment resulted in the total of 172 pages of submitted problems and answers that I posted a week prior to the final exam as extra practice (with solutions!). The submissions tend to be so rich in context and content that I now regularly incorporate them into next course offerings.

Compelling stories

There are many things that can make a story compelling. Structurally, one useful recipe is "Before and After": a description of a situation (this can be either a feeling, a situation or a statistic), followed by an event, resulting in a change of initial situation. Another way to look at it is the popular Challenge, Action, Result model – describe a challenging situation, subsequent action taken and causal outcome. It is preferable to have a "hook", so it certainly helps if the original situation is somewhat dire and the resolution is in the direction of a happy ending or the result is overall quite dramatic. One example from my practice is the story of Yellowstone wolves: going from complete wolf extirpation – which resulted in drastic decline in the park's condition, largely due to elk overpopulation – to eventual re-introduction of the wolves and the park's ecosystem restoring itself.

The problem's appeal can come from many places. Say, you found a great research paper about modeling the rate of an enzyme-catalyzed chemical reaction based on the concentration of the substrate. This would be a great application to introduce to your calculus section full of biology and chemistry majors. The good news is that it's a lovely rational function (known as Michaelis-Menten kinetics) that you can fully analyze using calculus tools developed in a standard first-term course. The bad news is now you have to explain in your math class what enzyme, catalyst, substrate and concentration is — some students might have seen it before, but surely not all of them are familiar with the terminology. So you will need to strike a careful balance between the non-mathematical jargon and the mathematical set up. My rule of thumb is 3 minutes – I should be able to cover non-mathematical motivation or necessary terminology in under 3 minutes and sometimes I have to carefully structure my use of time here. But in this case, more importantly, how can we make this seemingly dry problem appealing?

Sometimes what makes an application fascinating is not directly its mathematical or non-mathematical content, but rather the people behind it. Even math majors don't often know much about the fascinating humans behind mathematical advances. Stories about mathematicians as people – with their quirky personalities, with their human experiences, passions and flaws – highlight the development of mathematics as a uniquely human experience. So back to Michaelis-Menten kinetics. Maud Menten grew up in BC and is now buried there, she spoke several languages including Halq'em'eylem, went on an Arctic expedition, drove a Model T Ford. She boarded the boat alone in 1912 to cross the Atlantic Ocean to study enzyme-catalysed reactions – don't you want to find out what she discovered on the other side?

Most times, you only need to get a little bit creative to find a compelling exposition. I have to admit: I enjoy thinking about different shaped containers and boxes. Of course there are many standard calculus problems about minimizing surface area while maximizing volume, but those don't even excite me, much less my students. So how can I share my love of boxes with my students in an engaging way? I tell stories, of course. I share a story of square milk jugs introduced in the US. Previously, 1-gallon US milk jugs looked like the Canadian ones (those of you who buy milk in bags will have to look up the pictures). So why change the design? Discussion ensues… Turns out the new jugs are more environmentally friendly since they are stackable and eliminate the need for milk crates, which use a lot of water and chemicals to clean, they also break often and get stolen very often (who knew the milk jugs were a popular petty theft item?). Maybe they also use less material in production – that's where calculus comes in: we discuss simplifying assumptions regarding jugs' shapes and the Canadian version, being closer to the cube, wins in the smaller surface area competition. That's not where the story ends though; it turns out the square jugs caused quite a stir – consumers hated them as they were harder to pour milk out of and it turned out they use so much plastic wrap to keep them stacked that it eliminated the environmental benefits originally envisioned by their designers.

My favourite box application is inspired by Indigenous crafts local to where I live: Coast Salish bentwood boxes. These open top boxes are made out of a flat cedar plank that is steamed and bent to form a box. There are many questions you can ask right away: how do we need to bend the plank to create a rectangular shape, no cutting allowed? Where do we need to bend it to not waste any wood? Where do we need to bend it to create a box of largest volume? What if we also allow one cut to create a lid that fits? Mathematically, these are fairly standard questions though students generally struggle with 3-dimensional objects, so we always construct "bentpaper" boxes together. Contextually, students engage in this problem by exploring local traditions (what were the boxes used for?), materials (would any wood or plank do?) and crafts (how do we physically bend the wood?). The latter part emphasizes the simplifying assumptions we make to create a mathematical model: we assume that the thickness of the wood is negligible, while the craftsperson needs to account for it and carve a groove at the corners.

Finally, you cannot forget the effects of genuine excitement. My love for boxes does not go unnoticed: I often get several final projects submitted inside handmade boxes, one time I got a submission on a wood plank that I was encouraged to bend into a box, another time a student gave me a book on constructing origami boxes. My fascination with Maud Menten is also apparently contagious as last term one of my students went on a quest to visit places where she lived in the Fraser Valley and even found her tombstone there. Genuine love for stories produces genuine stories and genuine connections.

Complex, multifaceted problem domain

> "Alice: Would you tell me, please, which way I ought to go from here?
>
> The Cheshire Cat: That depends a good deal on where you want to get to."

We often tell the students that math is everywhere, but they can rarely recognize it because they do not know what to look for: there are no polynomials hanging out in mid-air and no ladders sliding down random walls. Their math always came from books with carefully constructed examples, targeting very specific concepts, with mostly integer solutions, with applications that are never personally relevant, relatable or significant enough. So students learn to ignore the make-up of the problems or really any words surrounding the formulas and practice choosing (or guessing) the correct algorithm to apply.

There is a Chekhov principle in writing: "If in the first act you have hung a pistol on the wall, then in the following one it should be fired", meaning that notable details should play an important part in the plot. I will lightly advocate against this principle here. While you shouldn't sprinkle random redundant information around, the problem setup should be rich enough to give rise to different questions to be asked, different reasonable things to optimize or solve for, different approaches to be utilized, different ways for the plot of the solution to develop. Contextually, the most impactful stories involve activating the imagination of the listener where they place themselves in the situation described in the story. In the Yellowstone wolves' story, students act as ecologists that need to argue for the wolf re-introduction into the park: they decide on what type of predator wolves are and hence what function to use to model their behavior, they decide on simplifying assumptions we are comfortable making, they decide how we will approximate elk reproduction rate, they choose what we would like to optimize or solve for. Having an entry point and having options within a problem gives everyone an opportunity to see themselves in it. I do have to adapt to each class's choices (small price to pay for high levels of engagement), but we also discuss how our decisions will affect the model outcome.

Sense making in context

I shot an elephant wearing my pyjamas. Was I wearing my pyjamas or was the elephant? Did I use a rifle or a camera? Imagine saying this sentence to an audience full of kids — to them an elephant in pink-striped pajamas is a completely plausible (and most definitely a more amusing) scenario. And if I am an ivory poacher bragging to my friends, then I'm surely not talking about my camera. Context matters and so do people's experiences within this context.

There are two main ways of building intelligence: giving the subject a recipe or giving them the opportunity to grow themselves. With the rise of the utility of Artificial Intelligence, I am reminded of its humble origins: the first version of IBM's Watson. The researchers found the hard way that giving anyone or anything a database of facts, no matter how large, won't get you very far — they loaded the machine with 15 terabytes of information, but without being able to *learn* from it, the first version of Watson was able to answer only about 10% of Jeopardy! questions correctly. The system builders needed to provide Watson with ways to link information together and to be able to place relevant facts in relevant context. Facts, rules and a lot of mathematical modeling did have its effect: Watson did eventually win Jeopardy! against human players and we now have many examples of practical and useful language models. But how do we teach our students that the facts alone won't do?

Authentic applications provide an easy entry point into this aspect of mathematical learning. Students often come to us with preconceived notions that any math application is based on artificial set up and hence its conclusions needn't make sense. In practice, bringing in real stories from external sources (news, reports, arts) tends to force the issue of interpretation making sense for the context at hand. Of course I still get occasional blunders where students do not perform a common sense check of their answer, but those are both rare and easy to question. In a sliding ladder problem, the answer can be almost anything without raising suspicion of its correctness; but if in a problem modeling fetus growth a student gets a negative rate of change, you can ask them if it is really feasible that a fetus body part starts shrinking.

But sense making is also a place to be humble. Students will offer entry points and interpretations of the problem that you didn't think of. They will offer answers that seem unreasonable under your assumptions, but perfectly plausible under theirs. Science in general and math in particular is not a transfer of knowledge from one person to another with a fixed set of axioms and limited options. Rather, it should be a rich and an integrated reflection of how the world functions.

> I shot an elephant wearing my pyjamas.

Problem centric

I generally believe in teaching **through** content, not **about** content. Whether the content is mathematically inspired, biologically grounded or physically established, it becomes the central motivation: start with a problem and develop tools to analyze and solve it, not the other way around. A problem-based approach can be implemented on a small scale or taken through the entire semester: the Yellowstone wolves' example above is a course-long problem, whereas the milk jug or bentwood boxes are examples of shorter problems. The goal is to make problem the central focus, not the afterthought; math arises and is developed from the questions we can ask in the context of the problem. The real power of math comes in, however, when we see that we can apply the same mathematical tools in different contexts: the construction and analysis of the rational functions performed in the Michaelis-Menten kinetics is the same one used to understand and model predator functional response.

Problems chosen can be rooted in any field, including mathematics itself (best suited for a math majors' course). For example, consider the brachistochrone problem posed by Johann Bernoulli in 1696 essentially as a challenge to Isaac Newton: *"Given two points A and B in a vertical plane, find the curve traced out by a point acted on only by gravity, which starts at A and reaches B in* *the shortest time."* This problem is easy to understand but provides a way to motivate the study of and access to various topics in calculus of variations including model construction, numerical solutions, Euler-Lagrange equation and functional integrals. How about the bicycle tracks problem here that comes from a Sherlock Holmes novel: given a set of bicycle tracks, can you tell which way the bike went? Here you really need to figure out which questions (of the tracks and bike construction in general) to even ask first.

Informed decision making, predictive ability, impactful outcome

We are constantly bombarded by data, charts, graphs and general numerical information in our daily lives. So there really is no excuse to not use real data sources in our teaching. Not only does it add authenticity to the classroom material, it also contributes to the creation of the more educated society, ensuring that our students are able to critically consider information presented to them, question assumptions and conclusions, evaluate the situation and make informed decisions about it themselves.

I find various pages on Statistics Canada and Statista useful for inspiration to build applications around. My students' learning journals also contain a ton of useful ideas and context examples from their other courses or extra curriculars. I often use local data to discuss issues of local importance and together with students investigate how math can help us in a variety of ways. We investigate fish stocking reports, learn some basic facts about the various animals and their feeding habits; we examine populations of Skeena salmon and talk about salmon life cycle as well as its importance to local communities; we study fascinating hunting habits of Northwest crows; we model selenium pollution of Lake Koocanusa. The data we use comes from government reports, independent studies, research papers, historical records.

Interacting with these problems is not just a simple find the answer and move on process. Even when all is set and done (ie we found the answer to the question we asked of the problem in the beginning), we need to interpret the mathematical results in the context of the problem: do the results make sense given the real-world scenario? Based on our mathematical analysis, what informed decisions or recommendations would we make? How would small changes in input parameters or assumptions affect the outcomes of our analysis? What are the limitations of our solution? How will we communicate these findings? What insights could the various numbers and graphs provide? Are there ways to refine the model, improve accuracy, incorporate additional factors? … Genuine problem solving process highlights the complexity of each situation and emphasizes that there likely is not one right answer. Our understanding of the world may be illuminated if we are willing to admit more than one truth.

Final thoughts

I am often asked where I find the context for my applications. I do browse aforementioned websites and look for something that appeals to me and that I would like to develop a story about. But also, I often stumble upon applications that I want to use. It's not just dumb luck: I stumble upon applications because the idea of developing problems to use out of something I witness is always in the back of my mind. Most recently, my daughter and I went to the dinosaur exhibit at the Canadian Museum of Nature, where we saw (among many dinosaur skeletons) the McNeill formula for relating the leg length with the walking speed of an animal; so naturally, in my calculus course, we took a closer looks at the several mathematical models proposed by Richard McNeill Alexander, who studied biomechanics of dinosaurs.

We tell our students that math is all around us, so let's start noticing it ourselves.

Endnotes

1. Viktor Blasjo on his podcast "Opinionated History of Mathematics" offers a wonderful insight into how the "pure" applications came to us from antiquity, I strongly recommend this episode: https://intellectualmathematics.com/blog/societal-role-of-geometry-in-early-civilisations/
2. Gloria Steinem writes the following story in her book "My Life on the Road": "I took a course in geology because I thought it was the easiest way of fulfilling a science requirement. One day the professor took us out into the Connecticut River Valley to show us the 'meander curves' of an age-old river. I was paying no attention because I had walked up a dirt path and found a big turtle, a giant mud turtle about two feet across, on the muddy embankment of an asphalt road. I was sure it was going to crawl onto the road and be crushed by a car. So with a lot of difficulty, I picked up the huge snapping turtle and slowly carried it down the road to the river. Just as I had slipped it into the water and was watching it swim away, my geology professor came up behind me. "You know," he said quietly, "that turtle has probably spent a month crawling up the dirt path to lay its eggs in the mud on the side of the road—you have just put it back in the river." I felt terrible. I couldn't believe what I had done, but it was too late. It took me many more years of organizing to realize that this parable had taught me the first rule of organizing "always ask the turtle".