The Surprising Role of the Pendulum in Stokes's Discovery of New Physical Theory and Use of Divergent Series

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Consider a function, represented in its infinite series Taylor expansion. For example:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

It is possible to approximate the value of e^x for any value of x by using the first n terms of the series and then truncating and ignoring the rest. This procedure produces an approximation with which an error is associated.

In the case of this example, the infinite series for e^x converges for all values of x, resulting in an approximation with increasingly less error as n increases. However, depending on the function and the representative infinite series, the series may converge for only limited values of the argument x. Convergent series are well understood, with well-defined methods of bounding the error of the approximations made by using truncated infinite series.

Divergent series, on the other hand, are more difficult and more interesting. A canonical example of a function and its associated divergent infinite series representation is the Stirling series for the factorial function,

$$n! \sim \sqrt{2\pi n}\left(\tfrac{n}{e}\right)^n \left(1 + \tfrac{1}{12n} + \tfrac{1}{288n^2} - \cdots\right).$$

The infinite series on the right diverges for all values of n. However, a good approximation for $n!$ can be obtained by using a carefully selected number of terms of the divergent series. Furthermore, even though the series is divergent, it is possible to bound, but not to make arbitrarily small, the error in the approximation.

The Stirling series representation for $n!$ was formulated by Abraham De Moivre (1667–1754) and improved upon by James Stirling (1672–1770). The series was shown to be divergent by Thomas Bayes (ca 1702–1761) in a letter published in 1763. This type of series, when truncated after a given number of terms, is now called an asymptotic expansion of the function, or an asymptotic approximation to the function. Nearly all mathematicians by 1800 were familiar with this example, and it caused confusion and consternation along with appreciation for its obvious utility.

The utility of asymptotic approximations during the 18th and early 19th centuries—for efficient computation, in particular—could not be ignored, particularly in an era when computations were done by hand. It was, however, not clear at this time why these approximations worked so well, or what the ultimate divergence of the series meant for the approximation.

Figure 1. George Gabriel Stokes (1819–1903). *Wikipedia*.

During the mid-19th century, divergent series were successfully used by George Gabriel Stokes (Figure 1) for two purposes. First, divergent series allowed Stokes to generate numbers from existing theory and then, second, he used those numbers to verify new physical theory. In trying to understand how divergent series were used in the mid-19th century and how the numbers generated by using them were verified, I became aware of the importance of the role of the pendulum.

In the early 19th century, in Britain and elsewhere, it was important to determine as accurately as possible the period of the pendulum, largely because pendulum measurements were extensively used in surveying, in navigation, and in the determination of physical constants, including the gravitational constant, the ellipticity of the Earth, and the mean density of the Earth.

The level of precision required meant that pendulum theory needed to move well beyond the idealized pendulum shown in Figure 2, in which simplifying assumptions causes significant differences between the calculated period of a pendulum and the actual period of a pendulum as measured in a laboratory.

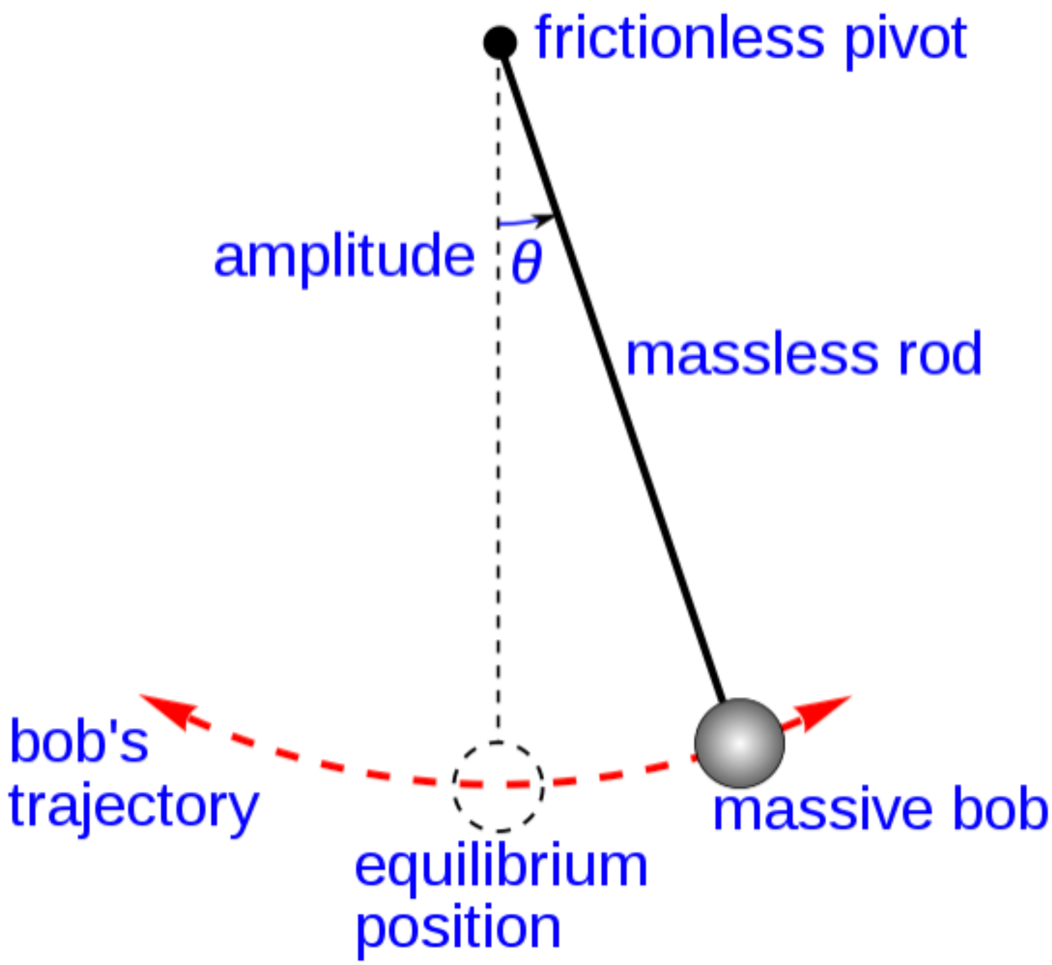

Figure 2. An idealized, mathematical model of the pendulum. *Wikipedia*.

An ideal pendulum consists of a pendulum bob swinging from a pivot on a rod. The pivot is considered to be frictionless, the rod is considered to be massless, and the pendulum bob is a massive point mass. Analyzing the forces on this idealized pendulum is a standard application of basic mechanics and yields an exact solution for the period of the pendulum in the form of an infinite series,

$$T = 2\pi\sqrt{\frac{L}{g}}\left(\sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}(n!)^2}\sin^{2n}\left(\frac{\theta_0}{2}\right)\right)$$

where the first few terms of the series are

$$T = 2\pi\sqrt{\frac{L}{g}}\left(1+\frac{1}{16}\theta_0^2+\frac{11}{3072}\theta_0^4+\dots\right).$$

If the assumption is made that the initial displacement angle, θ, is small, then the period of the pendulum is well approximated by the formula

$$T \approx 2\pi\sqrt{\frac{L}{g}}.$$

Error results from not taking into account the actual physical situation: e.g., there is friction at the pivot; the rod is not massless and its length may change with temperature or humidity; the pendulum bob is not a point mass and may not have uniform density, and it most certainly will be slowed by drag in the medium through which it is swung. In addition, temperature, altitude, and humidity change the drag characteristics of the medium through which the bob is swung.

A further source of error is the mathematical approximation made in order to make computation of the period possible. There is a loss of accuracy due to the truncation of the infinite series, and that error is a function of the initial displacement angle, which itself may be difficult to set repeatedly.

For the pendulum, as was typical practice, very precise empirical measurements were compared to predictions made from theory. Additionally, it was not only that theory was being validated by comparison to measured data. There was a further mathematical difficulty because it was a non-trivial task to produce numerical values from the theory for comparison in the first place, due to the prohibitively large number of calculations required.

The large number of computations resulted from the necessity of evaluating definite integrals that were not integrable in finite terms. Therefore, numerical results were being obtained via partial sums of convergent infinite series after integration of the power series of the integrand. However, the number of terms required to obtain the desired precision was too large for the human calculator.

By 1832, a large volume of very careful measurements of pendulum motion was available, and it was apparent to many natural philosophers that there was no theory available to explain the experimental results. These precise pendulum measurements led Stokes to new physical theory and, in order to generate the numbers from his theory, Stokes also developed asymptotic expansions in order to make possible the computations required.

These were both rather large breakthroughs. The first was the physical realization that there exists in fluid motion a previously unknown phenomenon—that which results in the boundary layer—which Stokes called the index of friction of the fluid. The second breakthrough was the ability to use asymptotic approximations to divergent series in order to obtain numbers from this new theory.

The new physical theory, in the case of the cylindrical bob, resulted in a convergent ascending series (meaning in increasing positive powers of the variable) involving the derivative of the gamma function. As noted above, the Stirling series, an asymptotic approximation to the gamma function, was stated long before Stokes's lifetime. It is possible that seeing the derivative of the gamma function caused Stokes to consider trying to find an analogous approximation method.

In the case of the cylinder, Stokes applied his new mathematical tool that consisted of converting the convergent series to a descending divergent series, from which he used the first few terms as an approximation. Stokes used the contemporaneous term "descending series" to refer to series where the power of the variable increases in the denominator as the terms progress. This simplified, in fact made possible, the numerical calculations. Stokes noted:

> The author has also obtained a descending series, which is much more convenient for numerical calculation when the diameter of the cylinder is large [1, p. 7].

In 1848 Stokes summarized the important formulae he had obtained, but he did not indicate how the formulae were obtained. He also provided three numerical calculations for three differently-sized cylindrical pendulum bobs. Stokes compared those calculations against previously-obtained standard theoretical results as well as with the experimental results. His new theory agreed much better with the experimental results than the earlier theoretical predictions, which did not account for internal friction, did. Here for the first time, as far as I know, Stokes announced his use of a divergent series for computation with respect to pendulums.

Pendulums were important for deducing results about physical phenomena and thus experimentalists such as Francis Baily (1774–1844) and Henry Kater (1777–1835) had spent a lot of time and energy in the early part of the 19th century on making accurate pendulum measurements, as Stokes was well aware:

> The great importance of the results obtained by means of the pendulum has induced philosophers to devote so much attention to the subject, and to perform the experiments with such a scrupulous regard to accuracy in every particular, that pendulum observations may be justly ranked among those most distinguished by modern exactness [3, p. 8].

A few pages later, as he prepared to discuss his investigations into the pendulum correction factors, Stokes described how the previous work had fallen short:

> The preceding [as I have summarized above] are all the investigations that have fallen under my notice, of which the object was to calculate from hydrodynamics the resistance to a body of given form oscillating as a pendulum. They all proceed on the ordinary equations of the motion of fluids. They all fail to account for one leading feature of the experimental results, namely, the increase of the factor n with a decrease in the dimensions of the body. They recognize no distinction between the action of different fluids, except what arises from their difference of density [3, p. 12].

None of the previous theories accounted for what is now termed viscosity, and it was evident from the experiments of several people that theory and experiment were not in agreement.

In contrast, Stokes produced a refined hydrodynamical theory that took into account viscosity, and that theory agreed with experimental evidence in the spherical pendulum case. No new mathematical techniques had been required for the computations thus far; those would be required for the cylindrical pendulum bobs.

Stokes then temporarily put aside the pendulum computations and attempted to compute zeros of the Airy integral. The theory for this was well-known—the difficulty lay entirely in being able to obtain numbers from the theory. After he devised a new mathematical technique that he used to make the Airy integral computations (which he verified against empirical data), Stokes returned to the pendulum computations in the case of the cylindrical bob and was able to use his new technique to obtain results:

> I found the method which I had employed in the case of this integral [the Airy integral] would apply to the problem of the resistance to a cylinder and it enabled me to get over the difficulty with which I had before beeen [*sic*] baffled. I immediately completed the numerical calculation so far as was requisite to compare the formulae with Baily's experiment on cylindrical rods, and found a remarkably close agreement between theory and observation [3, p. 13]

Stokes further verified his pendulum results by using an index of friction for water taken from the results of Coulomb's experiment on a spinning disk in water. (This value for the index of friction of water was not computed using a pendulum.) By using Coulomb's value for the index of friction to compute the pendulum vibration period of a pendulum swinging in water, Stokes found his theory agreed with experiments.

In addition to pendulum calculations, Stokes's discovery of the index of friction provided an explanation for the formation of clouds. In simplified terms, the explanation was as follows:

1. A sphere (water droplet) traveling uniformly in a fluid was considered as a limiting case of a ball pendulum as the length of the wire became arbitrarily large.
2. Stokes's theory showed that the resistance due to internal friction of a sphere moving through a fluid was proportional to the radius of the sphere rather than to the surface area of the sphere.
3. The index of friction for air was known from pendulum experiments.
4. The terminal velocity of the water droplets was calculated using the index of friction for air (the other sources of friction, proportional to the square of the velocity, were much less significant and were ignored) and was so small that the suspension of water droplets to form clouds was explained.

This was, perhaps, another verification of the new theory of the index of friction.

Stokes had developed a method, using divergent series, to obtain numbers from theory that were not previously practically computable, which he could then compare to something that was measurable in a laboratory. The pendulum had been a vital component in the advancement of theory and spurred the development of new mathematics. For further detail and contextualization, please see my recent thesis [1].

References

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[2] Stokes, George Gabriel (1848) On the resistance of the air to pendulums. *Notices and Abstracts of Miscellaneous Communications to the Sections*, 7–8. Appended to *Report of the Eighteenth Meeting of the British Association for the Advancement of Science; Held at Swansea in August 1848*, London: John Murray, 1849.

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Brenda Davison is a senior lecturer in mathematics at Simon Fraser University. Her academic interests include the history of mathematics and mathematics education; she is particularly interested in using the history of mathematics to enliven and inform the teaching of mathematics. Brenda obtained her PhD in pure mathematics in 2023 from Simon Fraser University with a dissertation titled "Divergent Series and Asymptotic Expansions, 1850–1900". Her avocations include reading literature, climbing, biking, and trail running.
