CMS NOTES

Just Not Obvious

Editorial

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You are probably familiar with Sylvanus Thompson's story about Lord Kelvin telling his physics class that Liouville was a ***mathematician***, that is to say, one to whom $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$ was as obvious as 2+2 would be to them. If you are also familiar with the devilishly clever trick with polar coordinates by which this integral is customarily evaluated, you will see that Kelvin was clearly using "obvious" in an after-the-fact sense: this is a fact that ***becomes*** obvious after exposition and a little thought. I fancy that not even the great Liouville would have "on-sighted" that problem! But perhaps Kelvin's use of the word is the more interesting one, applying as it does to more of a mathematician's interaction with the theorem in question.

I was at a very interesting colloquium talk a few weeks ago. The speaker, a member of our department, needed the Quadratic Reciprocity Theorem for something, and commented that while it was certainly true, and had been proved in a multitude of ways, none of them made the theorem obvious. I was glad to hear that — all these years, since I'd first met the QRT at Cambridge, I'd thought it was just me! But it got me thinking about what is and isn't obvious.

Sometimes it's just a matter of waiting for the right argument. The Sylvester-Gallai theorem, about configurations of lines and points in the Euclidean plane, was just "Sylvester's Problem" for almost fifty years after he posed it in 1893. In the middle of the twentieth century, proofs began to appear, culminating in Kelly's mininum-distance proof, which renders it truly obvious.

Some things in mathematics seem more obvious, on first acquaintance, than they really are. The four-color theorem seems obvious to many after a half-hour's doodling, but there's still no human-comprehensible proof. There are hard theorems about infinite sets whose finite counterparts are trivial. And what could be more obvious than the Jordan Curve Theorem, stating that every simple closed curve has an inside and an outside? But it's very, very difficult to prove, to the point that practically every undergraduate textbook that needs it punts on the proof. Why this discrepancy? I think it's because when you throw in a few extra "nice" properties the proof becomes very simple… and your imagination (or at least my imagination) tends to throw those nice properties in for free when you set yourself to "imagine a simple closed curve." Conversely, if we go up a few dimensions, n-spheres develop weird properties that are hard for our three-dimensional intuitions to imagine.

Then there are the different forms of the axiom of choice. What could be more obvious than that the product of a collection of nonempty sets is nonempty? But this is logically equivalent to the very nonobvious theorem of Tychonoff, and implies the bizarre Banach-Tarski dissection paradox, surely obvious to nobody.

Clearly we mean and understand something by "obvious" in mathematics… but it's not alway obvious what!