Peter Rosenthal Remembered

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Rather than giving technical details about Peter's many mathematical contributions, or listing his publications that can be easily accessed elsewhere, let me just talk a little about one central motivation for a great part of his research. Something that was the starting point for uncountable discussions, lectures, arguments, and even jokes with his close colleagues and collaborators, the group that I was so lucky to be a member of.

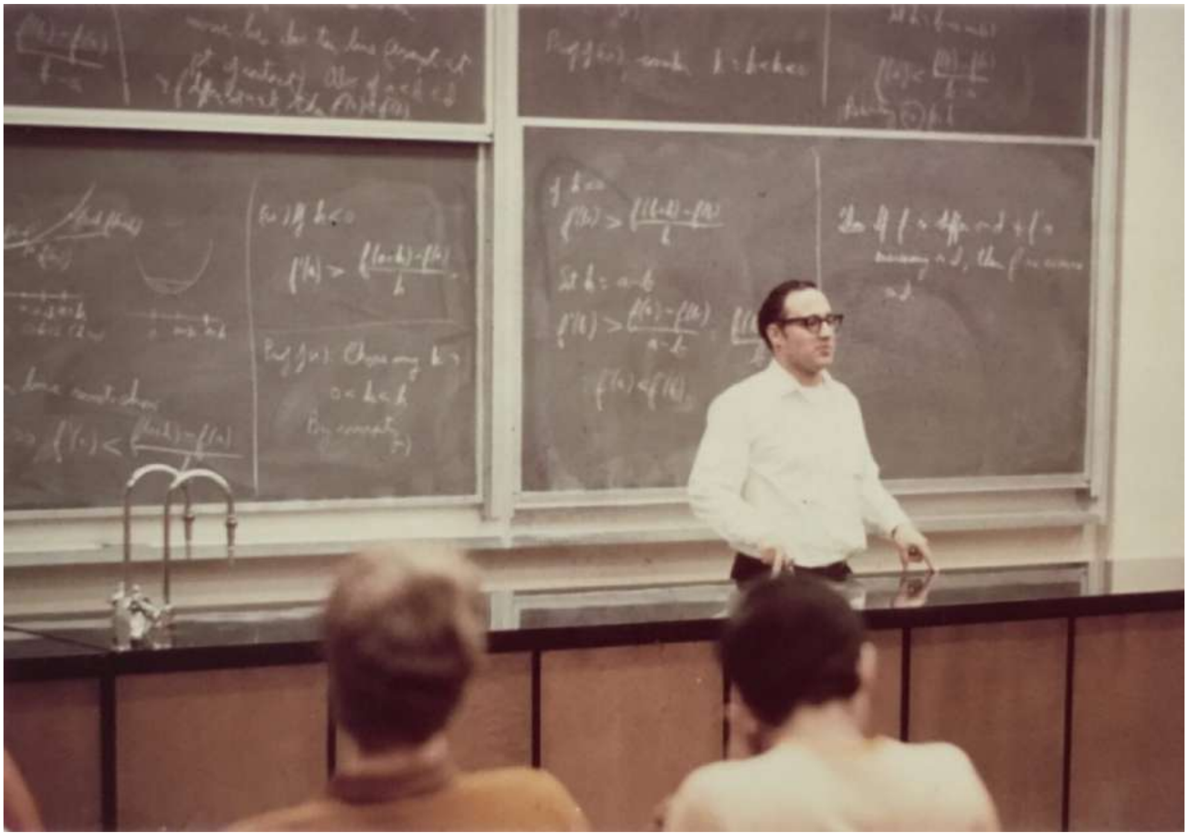
I am referring to the invariant subspace problem, a well known, and by now almost infamous, problem in operator-theory circles. It has been an unsolved problem about operators on a Hilbert space for many decades. (I should add here right away that Per Enflo has recently announced an affirmative solution to this problem and presented proofs, but his solution hasn't yet been verified and independently confirmed as I write this.)

This easily stated question is about the possible extension of the following simple fact in linear algebra: Every linear operator A on a finite-dimensional space H over complex numbers has a nontrivial invariant subspace; that is, there is a subspace K of H, other than {0} and H itself, which is taken into itself by A. (Okay, if you are going to be picky, we'll assume that H has dimension greater than one!) A more general statement is Burnside's theorem: the only transitive algebra of linear operators on a finite-dimensional complex space H is the algebra of all operators. "Transitive" means, of course, that the members of the algebra do not have a common nontrivial invariant subspace. (To see the existence of an invariant subspace for a single operator A, just note that the algebra generated by A is commutative, and if it were transitive, then, by Burnside, all linear operators would commute with each other; a contradiction.)

It was natural to ask whether these results could be generalized to infinite-dimensional settings. Continuous linear operators on Banach spaces, and particularly on Hilbert spaces, were considered. Let us emphasize here that when we say "subspace" now, we mean a closed subspace (which is automatic in finite dimensions, as is the continuity of our operators). Otherwise, the answer to the existence question is easily seen to be affirmative: Let A be any operator on the Hilbert space H. Pick a nonzero x in H and apply consecutive powers of A to x, take all linear combinations to form a subspace L of H, which is clearly of countable dimension, and thus a proper subspace of H, whose dimension is uncountable if infinite.

Negative results on a Banach space were obtained by Per Enflo and then C. J. Read, but the case of Hilbert spaces kept resisting a resolution. Various affirmative results were obtained, though always with additional hypotheses. For example, as every student of functional analysis learns early on, normal operators and compact operators have plenty of invariant subspaces.

A generalization of the Burnside theorem to the infinite-dimensional setting would be the following statement: an algebra of operators on H that don't have any common nontrivial invariant subspaces is dense in the whole algebra of operators on H. (The density here is in the appropriate sense of weak operator topology.) Again, all the affirmative results proven so far have used additional hypotheses. For example, W. B. Arveson proved it under the condition that the algebra contained a "substantial enough" normal operator. This is a natural generalization, to infinite dimensions, of what is a normal operator with distinct eigenvalues on a finite-dimensional space. Peter was instrumental in the wide ranging work that extended these results, by him and his students and collaborators. A sample result concerns algebras he called "Hermitian" first, and then "reductive" as suggested by P. R. Halmos. This means an algebra of operators on H with the property that the orthogonal complement of every (common) invariant subspace is also invariant. The extension was that a reductive algebra containing a substantial normal operator was dense in a selfadjoint algebra.



Just as Peter Rosenthal was writing his Ph.D. dissertation under Halmos's supervision at the University of Michigan, quite a few operator theorists besides Halmos were already busy with the above questions. Peter did very consequential work on the structure of lattices of invariant subspaces, and moved to the University of Toronto right after he got his degree. That is where I met him for the first time at the joint meeting of AMS and CMS in 1967. We were both admirers of Halmos and his approach to research and teaching. Our meeting started a long-lasting friendship and cooperation. The best and simplest way of describing his work with colleagues was that it was fun. For quite a few years Peter and I joined our colleagues Eric Nordgren and Don Hadwin at the University of New Hampshire every summer to work together. Every summer we started by declaring, half-seriously of course, that we would solve the invariant subspace problem by the end of the summer; then gradually settled to lighter goals. We also had various bets with each other and with other people on whether the problem had an affirmative solution. I remember one summer, when Peter came up with an idea for answering a question raised by C. Pearcy and A. Shields on how general Lomonosov's theorem was. Lomonosov had proved that every nonzero compact operator has a hyper-invariant subspace, i.e., the algebra of those operators that commute with it has a common nontrivial invariant subspace. The natural question that was then asked was, did this include all operators, perhaps? Peter suggested an excellent idea for a counterexample. It took some days, but it worked. Some bets were lost and won, but Peter was right: there was an operator A such that no non-scalar operator commuted both with it and with a nonzero compact operator.

After invariant subspaces what comes naturally is triangularization of operators, and then simultaneous triangularization of groups and semigroups of operators. Peter has made extensive contributions to research in this connection. I would like to mention one simple result here that geometers may find of some interest. It is a geometric equivalent of the invariant subspace problem, which does not even mention operators: Let H = K + L be a direct-sum decomposition of the Hilbert space H into two subspaces. Let us call a given subspace M of H "admissible" with respect to this decomposition if M is the sum of a subspace of K and a subspace of L. That is the only definition we need. Here comes a question equivalent to our stubborn one: If two different decompositions of H are given, does there exist a nontrivial subspace of H which is admissible with respect to both decompositions? That summer in New Hampshire we dreamed that a geometer smarter than us would come up with an answer.

Most readers of this short account are aware that Peter was well known, not just as a mathematician, but also as a lawyer, whose free services were often used in many legal cases in defence of marginalized clients. So, I will end with a related note. It was inevitable that Peter's mathematical colleagues would make remarks about the similarity of his life to Fermat's. They did, and he answered in kind. A gift I received from him about forty years ago is a T-shirt with a message. I bet you are in the process of guessing what the message said: "I have a proof of the invariant subspace problem. Shirt too small to put it on."

Peter will be missed by all who knew him.

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