

**Konstantin Guryev** (Simon Fraser University)

**Rina Zazkis** (Simon Fraser University)

*Education Notes bring mathematical and educational ideas forth to the CMS readership in a manner that promotes discussion of relevant topics including research, activities, issues, and noteworthy news items. Comments, suggestions, and submissions are welcome.*

**Egan J Chernoff**, University of Saskatchewan ([egan.chernoff@usask.ca](mailto:egan.chernoff@usask.ca))

**Kseniya Garaschuk**, University of the Fraser Valley ([kseiniya.garaschuk@ufv.ca](mailto:kseiniya.garaschuk@ufv.ca))

## The Procedural Trap vs. The Conceptual Reality

In mathematics classrooms in North America, an algebraic technique is deeply entrenched as the standard algorithm for finding an inverse function: “Switch  $x$  and  $y$ , then solve for  $y$ .” While algebraically efficient in simple, unconstrained linear cases, this mechanical routine frequently strips away the actual meaning of the function. It leaves both students and teachers disoriented when they encounter domain restrictions or real-world contexts where variables hold fixed meanings (Paoletti et al., 2018).

The conceptual reality is that an inverse function reverses the process of the original function – it maps outputs back to their original inputs. When instruction prioritizes procedural fluency over this logical foundation, the switching technique turns into a mathematical “trick” rather than a structural consequence. In fact, what escapes most students and teachers who rely of the swapping procedure is that it “works” based on the mathematical definition of an inverse.

Furthermore, this procedural focus often breeds a deeper cognitive avoidance of non-invertibility. For instance, when presented with a quadratic function over an unrestricted domain – such as  $f(x)=(x+1)^2$  for all real numbers – many students and teachers experience a strong resistance to declaring that “the inverse does not exist.” Instead of recognizing that the function fails injectivity, they often succumb to a “space of fuzziness,” providing a “slanted parabola” or writing down a flawed  $\pm$  square root expression (Marmur & Zazkis, 2018). Even those who recognize the issue often feel compelled to artificially restrict the domain on the spot just to avoid the claim of inexistence and revert to the known algorithm.

To explore what happens when a domain restriction is an inherent part of the problem, let us analyze a specific illustrative example that exposes the fragility of procedural automatism.

Consider the following problem: Find the inverse function of  $f(x)=(x+1)^2$  given the restricted domain  $x \leq -2$ .

Mathematically, there is no difference in the underlying logic whether one chooses to express the variable first or swap them first in case when variables hold no fixed meanings. However, the order in which a solver executes these procedures fundamentally alters the cognitive load and the visibility of the functional relationship. How do you expect your students will approach the task? For our students, the swapping approach appeared to be a preferred one, which often led them astray.

### Swapping $x$ and $y$ first (The Procedural Trap)

We have observed the cognitive stupor that frequently occurs when a student immediately applies the school curriculum convention of swapping the variables at the very first step:

$$x=(y+1)^2$$

The goal is now to isolate the “new”  $y$  and state the domain of the inverse. It is precisely here that the technique often breaks down into disconnected silos:

- **The Radical Stupor:** Upon trying to take the square root of both sides, some individuals freeze. They glance back at the original problem statement ( $x \leq -2$ ) and incorrectly deduce that they cannot take the square root of  $x$  because  $x$  is negative. They completely forget that the “new”  $x$  is actually the former  $y$  (the output), which is non-negative.
- **The Absolute Value Oversight:** Those who bypass this hurdle and write  $\sqrt{x} = y + 1$  routinely forget the absolute value entirely, or they assume it resolves with a positive sign because  $y+1$  “looks” positive. They fail to realize that because the “new”  $y$  is the former  $x$ , it is bound by the original constraint ( $y \leq -2$ ), making  $y+1$  negative. Thus, they miss the required negative sign, leading to an incorrect final function.

To avoid either pitfall, we describe a different approach.

### Expressing $x$ in terms of $y$ first (The Conceptual Path)

By maintaining the original variables during the algebraic manipulation, the functional roles of input and output remain clear and traceable:

1. We begin with the functional relationship:  $y=(x+1)^2$

2. Because both sides are non-negative within our constraints, we take the square root of both sides. Recalling that the square root of a square is the absolute value, we write:

$$\sqrt{y} = |x + 1|$$

3. Since the given domain states  $x \leq -2$ , it follows that  $x+1 \leq -1$ , meaning the expression inside the absolute value is negative. Thus, the absolute value clears with a negative sign:

$$\sqrt{y} = -(x + 1).$$

4. Solving for  $x$  yields:  $x = -\sqrt{y}-1$

At this stage, the mathematical inversion is entirely complete. We have successfully determined how the independent variable  $x$  depends on the output  $y$ . To finish the description of this new function, we identify its domain, which must coincide exactly with the range of the original function. Since  $x \leq -2$ , the output values are  $y \geq 1$ . Hence, the domain of the inverse is

$$y \geq 1.$$

For convenience, and to follow the standard convention that allows us to graph both functions on the same coordinate axes, we can now execute a variable swap:

$$y = -\sqrt{x-1} \text{ for } x \geq 1.$$

Every step in this progression preserves the logical connections between the functional law, the variables, and their corresponding domains.

### Conclusion

The “swap and solve” technique is incredibly powerful in unconstrained linear cases where students see an open path to isolating, giving them a safe sense of progress. However, as our quadratic example demonstrates, this structural safety is an illusion. In more complex or restricted settings, swapping the variables at the beginning can obscure the relationships among the function rule, the domain, and the range.

We admit that completely banishing a technique so deeply showcased in the classrooms may be impossible. Rather, teaching must shift instructional focus away from mechanical routines and toward meaningful solutions. Introductory calculus course in college or university may be an appropriate place to reconsider what “worked” in school. By anchoring procedural steps in their logical, structural foundations, we can ensure our students view algebraic manipulations not as isolated magic tricks, but as a direct reflection of the structural beauty of inverse relationships.

### References

Marmur, O. & Zazkis, R. (2018). Space of fuzziness: Avoidance of deterministic decisions in the case of the inverse function. *Educational Studies in Mathematics*, 99(3), 261-275.

Paoletti, T., Stevens, I. E., Hobson, N. L., Moore, K. C., & LaForest, K. R. (2018). Inverse function: Pre-service teachers' techniques and meanings. *Educational Studies in Mathematics*, 97(1), 93-109.

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**Canadian Mathematical Society** — 616 Cooper St., Ottawa, ON K1R 5J2, Canada